Lecture Notes in Differential Equations

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2011

Lecture Notes in Differential Equations
Lecture Notes to Accompany Math 280 & Math 351 (v. 12.0, 12/19/11)
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ISBN 978-0-557-82911-8
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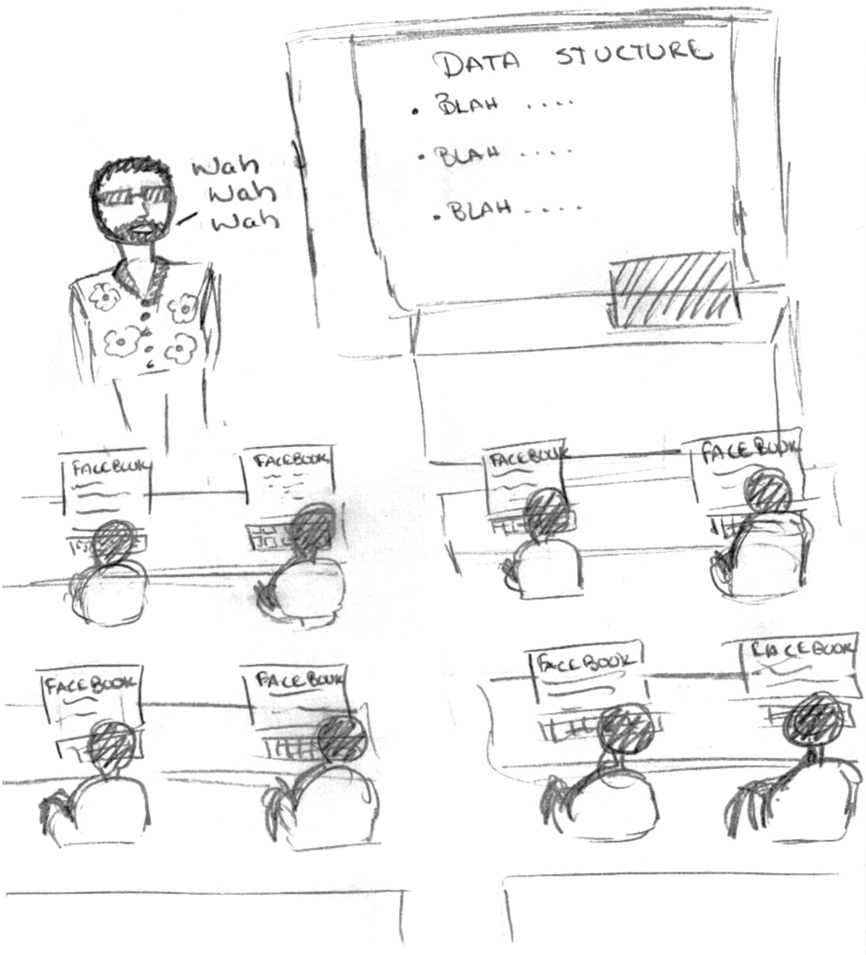
Dedicated to the hundreds of students who have sat patiently, rapt with attention, through my unrelenting lectures.



Preface

These lecture notes on differential equations are based on my experience teaching Math 280 and Math 351 at California State University, Northridge since 2000. The content of Math 280 is more applied (solving equations) and Math 351 is more theoretical (existence and uniqueness) but I have attempted to integrate the material together in the notes in a logical order and I select material from each section for each class.

The subject matter is classical differential equations and many of the exciting topics that could be covered in an introductory class, such as nonlinear systems analysis, bifurcations, chaos, delay equations, and difference equations are omitted in favor of providing a solid grounding the basics.

Some of the more theoretical sections have been marked with the traditional asterisk*. You can't possibly hope to cover everything in the notes in a single semester. If you are using these notes in a class you should use them in conjunction with one of the standard textbooks (such as [2], [9] or [12] for all students in both 280 and 351, and by [5] or [11] for the more theoretical classes such as 351) since the descriptions and justifications are necessarily brief, and there are no exercises.

The current version has been typeset in LaTeX and many pieces of it were converted using file conversion software to convert earlier versions from various other formats. This may have introduced as many errors as it saved in typing time. There are probably many more errors that haven't yet been caught so please let me know about them as you find them.

While this document is intended for students in my classes at CSUN you are free to use it and distribute it under the terms of the Creative Commons Attribution – Non-commercial – No Derivative Works 3.0 United States license. If you discover any bugs please let me know. All feedback, comments, suggestions for improvement, etc., are appreciated, especially if you've used these notes for a class, either at CSUN or elsewhere, from both instructors and students.

The art work on page i was drawn by D. Meza; on page iv by T. Adde; on page vi by R. Miranda; on page 10 by C. Roach; on page 116 by M. Ferreira; on page 204 by W. Jung; on page 282 by J. Peña; on page 330 by J. Guerrero-Gonzalez; on page 419 by S. Ross; and on page 421 by N. Joy. Additional submissions are always welcome and appreciated. The less humorous line drawings were all prepared by the author in Mathematica or Inkscape. □

Lesson 1

Basic Concepts

A differential equation is any equation that includes derivatives, such as

$$\frac{dy}{dt} = y \tag{1.1}$$

or

$$t^2 \frac{d^2y}{dt^2} + (1-t)\left(\frac{dy}{dt}\right)^2 = e^{ty} \tag{1.2}$$

There are two main classes of differential equations:

- **ordinary differential equations** (abbreviated **ODES** or **DES**) are equations that contain only ordinary derivatives; and
- **partial differential equations** (abbreviated **PDES**) are equations that contain partial derivatives, or combinations of partial and ordinary derivatives.

In your studies you may come across terms for other types of differential equations such as functional differential equations, delay equations, differential-algebraic equations, and so forth. In order to understand any of these more complicated types of equations (which we will not study this semester) one needs to completely understand the properties of equations that can be written in the form

$$\frac{dy}{dt} = f(t, y) \tag{1.3}$$

where $f(t, y)$ is some function of two variables. We will focus exclusively on equations of the form given by equation 1.3 and its generalizations to equations with higher order derivatives and systems of equations.

Ordinary differential equations are further classified by **type** and **degree**. There are two types of ODE:

- **Linear differential equations** are those that can be written in a form such as
$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_2(t)y'' + a_1(t)y' + a_0(t) = 0 \quad (1.4)$$
where each $a_i(t)$ is either zero, constant, or depends only on t, and not on y.

- **Nonlinear differential equations** are any equations that cannot be written in the above form. In particular, these include all equations that include y, y', y'', etc., raised to any power (such as y^2 or $(y')^3$); nonlinear functions of y or any derivative to any order (such as $sin(y)$ or e^{ty}; or any product or function of these.

The **order** of a differential equation is the degree of the highest order derivative in it. Thus
$$y''' - 3ty^2 = \sin t \quad (1.5)$$
is a third order (because of the y''') nonlinear (because of the y^2) differential equation. We will return to the concepts of degree and type of ODE later.

Definition 1.1 (**Standard Form**). A differential equation is said to be in standard form if we can solve for dy/dx, i.e., there exists some function $f(t, y)$ such that
$$\frac{dy}{dt} = f(t, y) \quad (1.6)$$

We will often want to rewrite a given equation in standard form so that we can identify the form of the function $f(t, y)$.

Example 1.1. Rewrite the differential equation $t^2y' + 3ty = yy'$ in standard form and identify the function $f(t, y)$.

The goal here is to solve for y':
$$\left.\begin{aligned} t^2y' - yy' &= -3ty \\ (t^2 - y)y' &= -3ty \\ y' &= \frac{3ty}{y - t^2} \end{aligned}\right\} \quad (1.7)$$
hence
$$f(t, y) = \frac{3ty}{y - t^2} \qquad \square \quad (1.8)$$

Definition 1.2 (Solution, ODE). A function $y = \phi(t)$ is called a solution of $y' = f(t, y)$ if it satisfies

$$\phi'(t) = f(t, \phi(t)) \tag{1.9}$$

By a **solution** of a differential equation we mean a function $y(t)$ that satisfies equation 1.3. We use the symbol $\phi(t)$ instead of $f(t)$ for the solution because f is **always** reserved for the function on the right-hand side of 1.3.

To **verify** that a function $y = f(t)$ is a solution of the ODE, is a solution, we substitute the function into both sides of the differential equation.

Example 1.2. A solution of

$$\frac{dy}{dt} = 3t \tag{1.10}$$

is

$$y = \frac{3}{2}t^2 \tag{1.11}$$

We use the expression "a solution" rather than "the solution" because solutions are not unique! For example,

$$y = \frac{3}{2}t^2 + 27 \tag{1.12}$$

is also a solution of $y' = 3t$. We say that the solution is **not unique**. □

Example 1.3. Show[1] that $y = x^4/16$ is a solution of $y' = xy^{1/2}$

Example 1.4. Show[2] that $y = xe^x$ is a solution of $y'' - 2y' + y = 0$.

Example 1.5. We can derive a solution of the differential equation

$$\frac{dy}{dt} = y \tag{1.13}$$

by rewriting it as

$$\frac{dy}{y} = dt \tag{1.14}$$

and then integrating both sides of the equation:

$$\int \frac{dy}{y} = \int dt \tag{1.15}$$

[1] Zill example 1.1.1(a)
[2] Zill example 1.1.1(b)

From our study of calculus we know that

$$\int \frac{dy}{y} = \ln|y| + C \tag{1.16}$$

and

$$\int dt = t + C \tag{1.17}$$

where the C's in the last two equations are *possibly different* numbers. We can write this as

$$\ln|y| + C_1 = t + C_2 \tag{1.18}$$

or

$$\ln|y| = t + C_3 \tag{1.19}$$

where $C_3 = C_2 - C_1$.

In general when we have arbitrary constants added, subtracted, multiplied or divided by one another we will get another constant and we will not distinguish between these; instead we will just write

$$\ln|y| = t + C \tag{1.20}$$

It is usually nice to be able to solve for y (although most of the time we won't be able to do this). In this case we know from the properties of logarithms that a

$$|y| = e^{t+C} = e^C e^t \tag{1.21}$$

Since an exponential of a constant is a constant, we normally just replace e^C with C, always keeping in mind that that C values are probably different:

$$|y| = Ce^t \tag{1.22}$$

We still have not solved for y; to do this we need to recall the definition of absolute value:

$$|y| = \begin{cases} y & \text{if } y \geq 0 \\ -y & \text{if } y < 0 \end{cases} \tag{1.23}$$

Thus we can write

$$y = \begin{cases} Ce^t & \text{if } y \geq 0 \\ -Ce^t & \text{if } y < 0 \end{cases} \tag{1.24}$$

But both C and $-C$ are constants, and so we can write this more generally as

$$y = Ce^t \tag{1.25}$$

So what is the difference between equations 1.22 and 1.25? In the first case we have an absolute value, which is never negative, so the C in equation

1.22 is restricted to being a positive number or zero. in the second case (equation 1.25) there is no such restriction on C, and it is allowed to take on any real value. □

In the previous example we say that $y = Ce^t$, where C is **any arbitrary constant** is the **general solution** of the differential equation. A constant like C that is allowed to take on multiple values in an equation is sometimes called a **parameter**, and in this jargon we will sometimes say that $y = Ce^t$ represents the **one-parameter family of solutions** (these are sometimes also called the **integral curves** or **solution curves**) of the differential equation, with parameter C. We will pin the value of the parameter down more firmly in terms of initial value problems, which associate a specific point, or **initial condition**, with a differential equation. We will return to the concept of the one-parameter family of solutions in the next section, where it provides us a geometric illustration of the concept of a differential equation as a description of a dynamical system.

Definition 1.3 (**Initial Value Problem (IVP)**)**.** An initial value problem is given by

$$\frac{dy}{dt} = f(t, y) \tag{1.26}$$

$$y(t_0) = y_0 \tag{1.27}$$

where (t_0, y_0) be a point in the domain of $f(t, y)$. Equation 1.27 is called an **initial condition** for the initial value problem.

Example 1.6. The following is an initial value problem:

$$\left.\begin{aligned} \frac{dy}{dt} &= 3t \\ y(0) &= 27 \end{aligned}\right\} \tag{1.28}$$

□

Definition 1.4 (**Solution, IVP**)**.** The function $\phi(t)$ is called a solution of the initial value problem

$$\left.\begin{aligned} \frac{dy}{dt} &= f(t, y) \\ y(t_0) &= y_0 \end{aligned}\right\} \tag{1.29}$$

if $\phi(t)$ satisfies both the ODE and the IC, i.e., $d\phi/dt = f(t, \phi(t))$ and $\phi(t_0) = y_0$.

Example 1.7. The solution of the IVP given by example 1.6 is given by equation 1.12, which you should verify. In fact, this solution is **unique**, in the sense that it is the only function that satisfies both the differential equation and the initial value problem. □

Example 1.8. Solve the initial value problem $dy/dt = t/y$, $y(1) = 2$.

We can rewrite the differential equation as

$$ydy = tdt \tag{1.30}$$

and then integrate,

$$\int ydy = \int tdt \tag{1.31}$$

$$\frac{1}{2}y^2 = \frac{1}{y}t^2 + C \tag{1.32}$$

When we substitute the initial condition (that $y = 2$ when $t = 1$) into the general solution, we obtain

$$\frac{1}{2}(2)^2 = \frac{1}{2}(1)^2 + C \tag{1.33}$$

Hence $C = 3/2$. Substituting back into equation 1.32 and multiplying through by 2,

$$y^2 = t^2 + 3 \tag{1.34}$$

Taking square roots,

$$y = \sqrt{t^2 + 3} \tag{1.35}$$

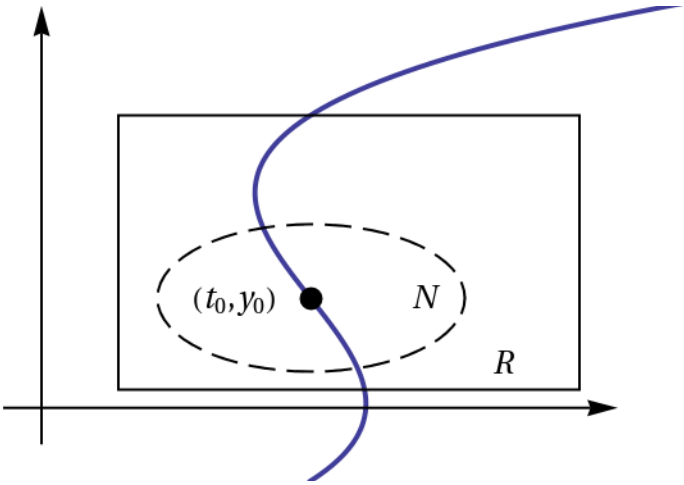
which we call *the* solution of the initial value problem. The negative square root is excluded because of the initial condition which forces $y(1) = 2$. □

Not all initial value problems have solutions. However, there are a large class of IVPs that do have solution. In particular, those equations for which the right hand side is differentiable with respect to y and the partial derivative is bounded. This is because of the following theorem which we will accept without proof for now.

Theorem 1.5 (**Fundamental Existence and Uniqueness Theorem**). Let $f(t, y)$ be bounded, continuous and differentiable on some neighborhood R of (t_0, y_0), and suppose that $\partial f/\partial y$ is bounded on R. Then the initial value problem 1.29 has a unique solution on some open neighborhood of (t_0, y_0).

Figure 1.1 illustrates what this means. The initial condition is given by the point (t_0, y_0) (the horizontal axis is the t-axis; the vertical axis is y). If there is some number M such that $|\partial f/\partial y| < M$ everywhere in the box R, then there is some region N where we can draw the curve through (t_0, y_0). This curve is the solution of the IVP.[3]

Figure 1.1: Illustration of the fundamental existence theorem. If f is bounded, continuous and differentiable in some neighborhood R of (t_0, R_0), and the partial derivative $\partial f \partial y$ is also bounded, then there is some (possibly smaller) neighborhood of (t_0, R_0) through which a unique solution to the initial value problem, with the solution passing through (t_0, y_0), exists. This does not mean we are able to find a formula for the solution.



A solution may be either **implicit** or **explicit**. A solution $y = \phi(t)$ is said to be **explicit** if the dependent variable (y in this case) can be written explicitly as a function of the independent variable (t, in this case). A relationship $F(t, y) = 0$ is said to represent and **implicit solution** of the differential equation on some interval I if there some function $\phi(t)$ such that $F(t, \phi(t)) = 0$ and the relationship $F(t, y) = 0$ satisfies the differential equation. For example, equation 1.34 represents the solution of $\{dy/dt = t/y,\ y(1) = 2\}$ implicitly on the interval $I = [-\sqrt{3}, \sqrt{3}]$ which (1.35) is an explicit solution of the same initial value problem problem.

[3]We also require that $|f| < M$ everywhere on R and that f be continuous and differentiable.

Example 1.9. Show that $y = e^{xy}$ is an implicit solution of

$$\frac{dy}{dx} = \frac{y^2}{1 - xy} \tag{1.36}$$

To verify that y is an implicit solution (it cannot be an explicit solution because it is not written as y as a function of x), we differentiate:

$$\begin{aligned}
\frac{dy}{dx} &= e^{xy} \times \frac{d}{dx}(xy) && (1.37)\\
&= y\left(x\frac{dy}{dx} + y\right) \qquad (\text{subst.} y' = e^{xy}) && (1.38)\\
&= yx\frac{dy}{dx} + y^2 && (1.39)\\
\frac{dy}{dx} - yx\frac{dy}{dx} &= y^2 && (1.40)\\
\frac{dy}{dx}(1 - yx) &= y^2 && (1.41)\\
\frac{dy}{dx} &= \frac{y^2}{1 - yx} \qquad \square && (1.42)
\end{aligned}$$

Definition 1.6 (Order). The order (sometimes called degree) of a differential equation is the order of the highest order derivative in the equation.

Example 1.10. The equation

$$\left(\frac{dy}{dt}\right)^3 + 3t = y/t \tag{1.43}$$

is first order, because the only derivative is dy/dt, and the equation

$$ty'' + 4y' + y = -5t^2 \tag{1.44}$$

is second order because it has a second derivative in the first term.

Definition 1.7 (Linear Equations). A linear differential equation is a DE that only contains terms that are linear in y and its derivatives to all orders. The linearity of t does not matter. The equation

$$y + 5y' + 17t^2y'' = \sin t \tag{1.45}$$

is linear but the following equations are not linear:

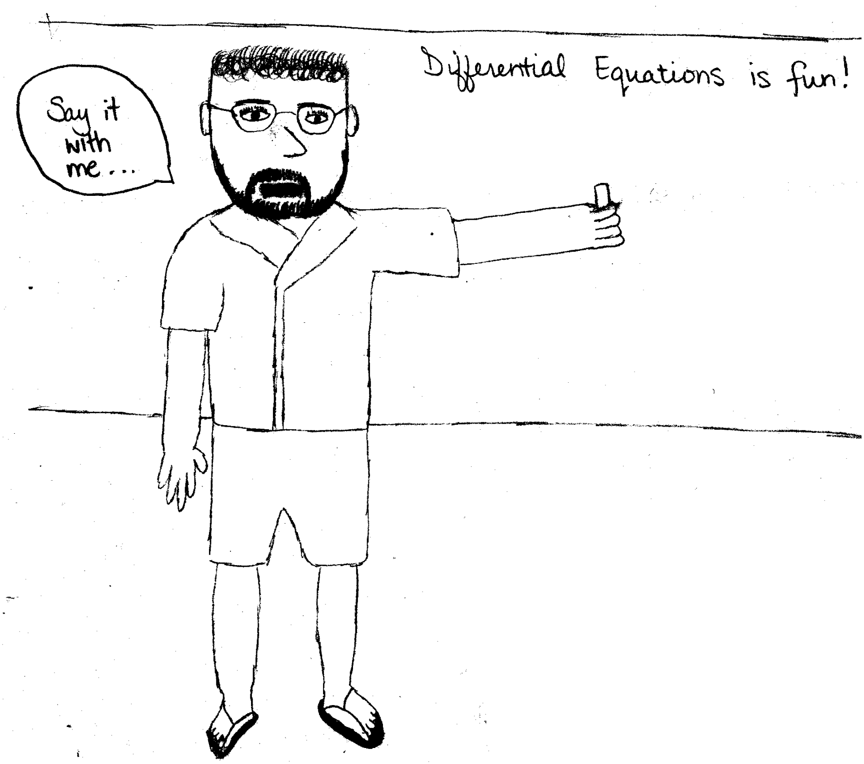
$$\begin{array}{ll}
y + 5t^2 \sin y = y'' & (\text{because of } \sin y)\\
y' + ty'' + y = y^2 & (\text{because of } y^2)\\
yy' = 5t & (\text{because of } yy')
\end{array} \tag{1.46}$$

We will study linear equations in greater detail in section 4.

Often we will be faced with a problem whose description requires not one, but two, or even more, differential equations. This is analogous to an algebra problem that requires us to solve multiple equations in multiple unknowns. A **system of differential equations** is a collection of related differential equations that have multiple unknowns. For example, the variable $y(t)$ might depend not only on t and $y(t)$ but also on a second variable $z(t)$, that in turn depends on $y(t)$. For example, this is a system of differential equations of two variables y and z (with independent variable t):

$$\left.\begin{aligned} \frac{dy}{dt} &= 3y + t^2 \sin z \\ \frac{dz}{dt} &= y - z \end{aligned}\right\} \tag{1.47}$$

It is because of systems that we will use the variable t rather than x for the horizontal (time) axis in our study of single ODEs. This way we can have a natural progression of variables $x(t)$, $y(t)$, $z(t)$, ..., in which to express systems of equations. In fact, systems of equations can be quite difficult to solve and often lead to chaotic solutions. We will return to a study of systems of linear equations in a later section.


Say it with me...
Differential Equations is fun!

Lesson 2

A Geometric View

One way to look at a differential equation is as a description of a trajectory or position of an object over time. We will steal the term "particle" from physics for this idea. By a particle we will mean a "thing" or "object" (but doesn't sound quite so coarse) whose location at time $t = t_0$ is given by

$$y = y_0 \tag{2.1}$$

At a later time $t > t_0$ we will describe the position by a function

$$y = \phi(t) \tag{2.2}$$

which we will generally write as $y(t)$ to avoid the confusion caused by the extra Greek symbol.[1] We can illustrate this in the following example.

Example 2.1. Find $y(t)$ for all $t > 0$ if $dy/dt = y$ and $y(0) = 1$.

In example 1.5 we found that the general solution of the *differential equation* is

$$y = Ce^t \tag{2.3}$$

We can determine the value of C from the initial condition, which tells us that $y = 1$ when $t = 1$:

$$1 = y(0) = Ce^0 = C \tag{2.4}$$

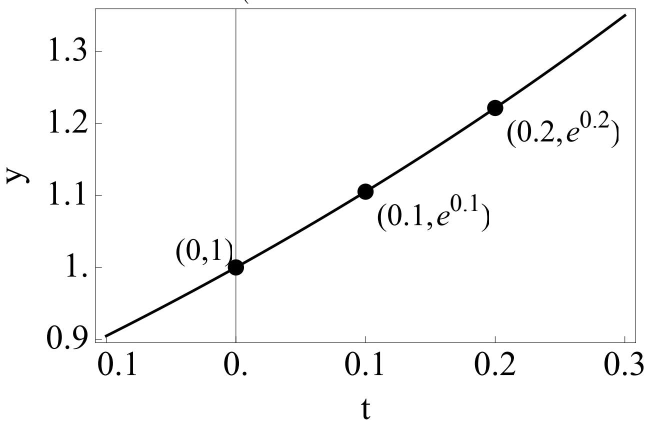
Hence the solution of the initial value problem is

$$y = e^t \tag{2.5}$$

[1] Mathematically, we mean that $\phi(t)$ is a solution of the equations that describes what happens to y as a result of some differential equation $dy/dt = f(t, y)$; in practice, the equation for $\phi(t)$ is identical to the equation for $y(t)$ and the distinction can be ignored.

We can plug in numbers to get the position of our "particle" at any time t: At $t = 0$, $y = e^0 = 1$; at $t = 0.1$, $y = e^(0.1) \approx= 1.10517$; at $t = 0.2$, $y = e^{0.2} \approx 1.2214$; etc. The corresponding "trajectory" is plotted in the figure 2.1. $\square$

Figure 2.1: Solution for example 2.1. Here the y axis gives the particle position as a function of time (the t or horizontal axis.



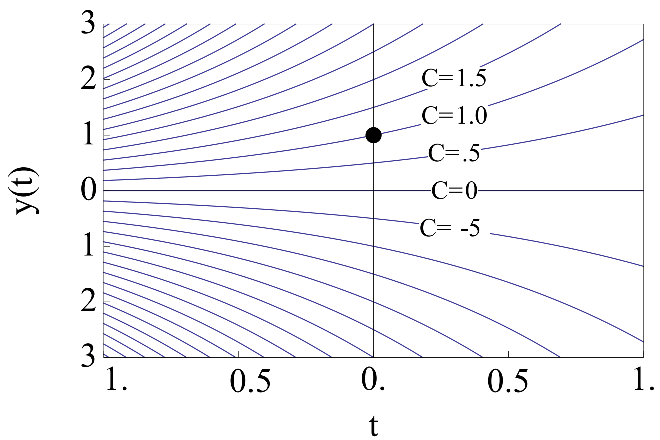
Since the solution of any (solvable[2]) initial value problem $dy/dt = f(t, y)$, $y(t_0) = y_0$ is given by some function $y = y(t)$, and because any function $y = y(t)$ can be interpreted as a trajectory, this tells us that any initial value problem can be interpreted geometrically in terms of a dynamical (moving or changing) system.[3] We say "geometrically" rather than "physically" because the dynamics may not follow the standard laws of physics (things like $F = ma$) but instead follow the rules defined by a differential equation. The geometric (or dynamic) interpretation of the initial value problem $y' = y$, $y(0) = 1$ given in the last example is is described by the plot of the trajectory (curve) of $y(t)$ as a function of t.

[2]By *solvable* we mean any IVP for which a solution exists according to the fundamental existence theorem (theorem 1.5). This does not necessarily mean that we can actually *solve for* (find) an equation for the solution.

[3]We will use the word "dynamics" in the sense that it is meant in mathematics and not in physics. In math a **dynamical system** is anything that is changing in time, hence dynamic. This often (though not always) means that it is governed by a differential equation. It does not have to follow the rules of Newtonian mechanics. The term "dynamical system" is frequently bandied about in conjunction with chaotic systems and chaos, but chaotic systems are only one type of dynamics. We will not study chaotic systems in this class but all of the systems we study can be considered dynamical systems.

We can extend this geometric interpretation from the initial value problem to the general solution of the differential equation. For example we found in example 1.5 that $y = Ce^t$ is the solution to the ODE $y' = y$, and we called the expression the **one-parameter family of solutions**. To see what this means consider the effect of the initial condition on $y = Ce^t$: it determines a specific value for C. In fact, if we were to plot every conceivable curve $y = Ce^t$ (for every value of C), the picture would look something like those shown in figure 2.2 The large black dot indicates the location of the point

Figure 2.2: Illustration of the one parameter family of solutions found in example 1.5.



(0, 1), and the values of C are shown for several of the curves.[4] We see that the curve corresponding to $C = 1.0$ is the only curve that passes through the point $(0, 1)$ - this is a result of the uniqueness of the solutions. As long the conditions of the fundamental theorem (theorem 1.5) are met, there is always precisely one curve that passes through any given point. The *family* (or collection) of curves that we see in this picture represents the one-parameter family of solutions: each member of the family is a different curve, and is differentiated by its relatives by the value of C, which we call a parameter. Another term that is sometimes used for the one-parameter family of solutions is the set of **integral curves**.

[4]In fact, not all curves are shown here, only the curves for $C = \dots, -.5, 0, 0.5, 1, 1.5, \dots$. Curves for other values fall between these.

Example 2.2. Find and illustrate the one-parameter family of solutions for the ODE

$$\frac{dy}{dt} = -\frac{t}{y} \tag{2.6}$$

Cross multiplying and integrating

$$\int y dy = -\int t dt \tag{2.7}$$

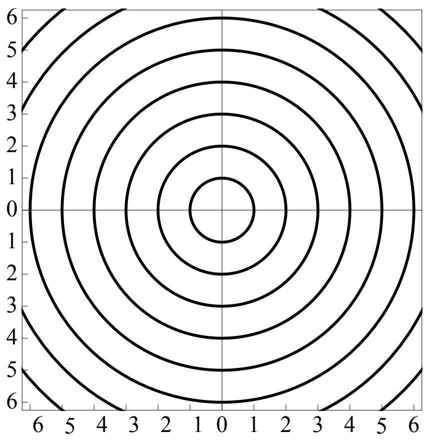
$$\frac{1}{2}y^2 = -\frac{1}{2}t^2 + C \tag{2.8}$$

Multiplying through by 2, bringing the t to the left hand side and redefining $C' = 2C$ gives us

$$y^2 + t^2 = C^2 \tag{2.9}$$

which we (should!) recognize as the equation of a circle of radius C. The curves for several values of $C = 1, 2, \ldots$ is illustrated in figure 2.3. $\square$

Figure 2.3: One parameter family of solutions for example 2.2.



Sometimes its not so easy to visualize the trajectories; a tool that gives us some help here is the **direction field**. The direction field is a plot of the slope of the trajectory. We know from the differential equation

$$\frac{dy}{dt} = f(t, y) \tag{2.10}$$

that since the slope of the solution $y(t)$ at any point is dy/dt, and since $dy/dt = f$, then the slope at (t, y) must be equal to $f(t, y)$. We obtain the direction field but dividing the plane into a fixed grid of points $P_i = (t_i, y_i)$ and then then drawing a little arrow at each point P_i with slope $f(t_i, y_i)$. The lengths of all the arrows should be the same. The general principal is illustrated by the following example.

Example 2.3. Construct the direction field of

$$\frac{dy}{dt} = t^2 - y \tag{2.11}$$

on the region $-3 \le t \le 3, -3 \le y \le 3$, with a grid spacing of 1.

First we calculate the values of the slopes at different points. The slope is given by $f(t, y) = t^2 - y$. Several values are shown.

t	$y=-3$	$y=-2$	$y=-1$	$y=0$	$y=1$	$y=2$	$y=3$
$t=-3$	12	11	10	9	8	7	6
$t=-2$	7	6	5	4	3	2	1
$t=-1$	4	3	2	1	0	-1	-2
$t=0$	3	2	1	0	-1	-2	-3
$t=1$	4	3	2	1	0	-1	-2
$t=2$	7	6	5	4	3	2	1
$t=3$	12	11	10	9	8	7	6

The direction field with a grid spacing of 1, as calculated in the table above, is shown in figure 2.4 on the left.[5] At each point, a small arrow is plotted. For example, an arrow with slope 6 is drawn centered on the point (-3,3); an arrow with slope 1 is drawn centered on the point (-2, 3); and so forth. (The values are taken directly with the table). A direction field of the same differential equation but with a finer spacing is illustrated in fig 2.4 on the right. From this plot we can image what the solutions may look like by constructing curves in our minds that are tangent to each arrow.

Usually it is easier if we omit the arrows and just draw short lines on the grid (see figure 2.5, on the left[6]). The one-parameter family of solutions is illustrated on the right.[7] □

[5]The direction field can be plotted in Mathematica using `f[t_, y_] := t^2 - y; VectorPlot[{1, f[t, y]}/Norm[{1, f[t, y]}], {t, -3, 3}, {y, -3, 3}]`. The normalization ensures that all arrows have the same length.

[6]In Mathematica: `f[t_, y_] := t^2 - y;` followed by `v[t_, y_, f_] := 0.1*{Cos[ArcTan[f[t, y]]], Sin[ArcTan[f[t, y]]]}; L[t_, y_, f_] := Line[{{t, y - v[t, y, f], {t, y} + v[t, y, f]}]; Graphics[Table[L[t, y, f], {t, -3, 3, .2}, {y, -3, 3, .2}]]`

[7]The analytic solution $y = e^{-t}\left(e^t t^2 - 2e^t t + 2e^t - e^{t_0} t_0{}^2 + e^{t_0} y_0 - 2e^{t_0} + 2e^{t_0} t_0\right)$ was used to generate this plot. The solution can be found in Mathematica via `DSolve[{y'[t] == t^2 - y[t], y[t0] == y0}, y[t], t]`.

Figure 2.4: Direction fields with arrows. See text for details.

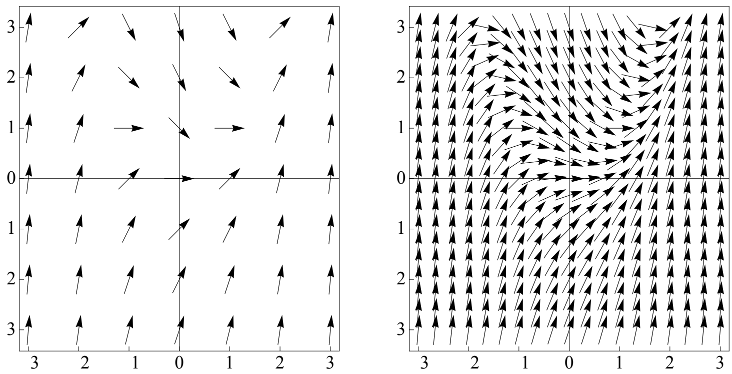
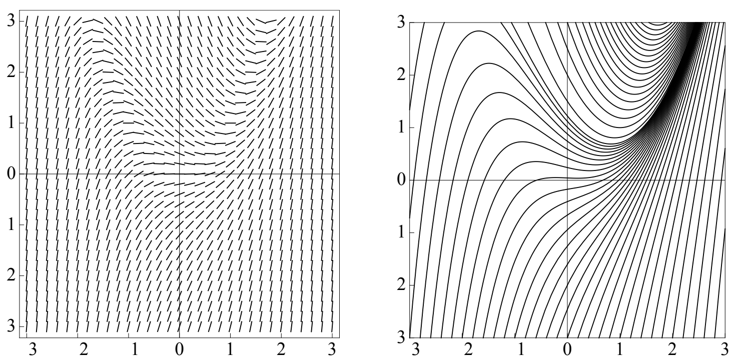


Figure 2.5: Direction fields with short lines (left) and one parameter family of solutions (right) for (2.11).



Lesson 3

Separable Equations

An ODE is said to be **separable** if the parts that depend on t and y can be separated to the different sides of the equation. This makes it possible to integrate each side separable.[1] Specifically, an equation is separable if it can be written is

$$\frac{dy}{dt} = a(t)b(y) \tag{3.1}$$

for some function $a(t)$ that depends only on t, but not on y, and some function $b(y)$ that depends only on y and and not on t. If we multiply through by dt and divide by $b(y)$ the equation becomes

$$\frac{dy}{b(y)} = a(t)dt \tag{3.2}$$

so we may integrate:

$$\int \frac{dy}{b(y)} = \int a(t)dt \tag{3.3}$$

We have already seen many separable equations. Another is given in the following example.

Example 3.1.

$$\left.\begin{aligned} \frac{dy}{dt} &= \left(1 - \frac{t}{2}\right) y^2 \\ y(0) &= 1 \end{aligned}\right\} \tag{3.4}$$

[1] In this section of the text, Boyce and DiPrima have chosen to use x rather than t as the independent variable, probably because it will look more like exact two-dimensional derivatives of the type you should have seen in Math 250.

This equation is separable with

$$a(t) = 1 - \frac{t}{2} \tag{3.5}$$

and

$$b(y) = y^2 \tag{3.6}$$

and it can be rewritten as

$$\frac{dy}{y^2} = \left(1 - \frac{t}{2}\right) dt \tag{3.7}$$

Integrating,

$$\int y^{-2} dy = \int \left(1 - \frac{t}{2}\right) dt \tag{3.8}$$

$$-\frac{1}{y} = t - \frac{1}{4}t^2 + C \tag{3.9}$$

The initial condition gives

$$-1 = 0 - 0 + C \tag{3.10}$$

hence

$$\frac{1}{y} = \frac{1}{4}t^2 - t + 1 = \frac{1}{4}(t^2 - 4t + 4) = \frac{1}{4}(t-2)^2 \tag{3.11}$$

Solving for y,

$$y = \frac{4}{(t-2)^2} \quad \square \tag{3.12}$$

Often with separable equations we will not be able to find an explicit expression for y as a function of t; instead, we will have to be happy with an equation that relates the two variables.

Example 3.2. Find a general solution of

$$\frac{dy}{dt} = \frac{t}{e^y - 2y} \tag{3.13}$$

This can be rearranged as

$$(e^y - 2y)dy = tdt \tag{3.14}$$

Integrating,

$$\int (e^y - 2y)dy = \int tdt \tag{3.15}$$

$$e^y - y^2 = \frac{1}{2}t^2 + C \tag{3.16}$$

Since it is not possible to solve this equation for y as a function of t, is is common to rearrange it as a function equal to a constant:

$$e^y - y^2 - \frac{1}{2}t^2 = C \quad \square \tag{3.17}$$

Sometimes this inability to find an explicit formula for $y(t)$ means that the relationship between y and t is not a function, but is instead multi-valued, as in the following example where we can use our knowledge of analytic geometry to learn more about the solutions.

Example 3.3. Find the general solution of

$$\frac{dy}{dt} = -\frac{4t}{9y} \tag{3.18}$$

Rearranging and integrating:

$$\int 9y dy = -\int 4t dt \tag{3.19}$$

$$\frac{9}{2}y^2 = -2t^2 + C \tag{3.20}$$

$$9y^2 + 4t^2 = C \quad \text{(Different } C\text{)} \tag{3.21}$$

Dividing both sides by 36,

$$\frac{y^2}{4} + \frac{t^2}{9} = C \quad \text{(Different } C\text{)} \tag{3.22}$$

Dividing by C

$$\frac{y^2}{4C} + \frac{t^2}{9C} = 1 \tag{3.23}$$

which is the general form of an ellipse with axis $2\sqrt{C}$ parallel to the y axis and axis $3\sqrt{C}$ parallel to the y axis. Thus the solutions are all ellipses around the origin, and these cannot be solved explicitly as a function because the formula is multi-valued. $\square$

Example 3.4. Solve[2] $(1 + x)dy - ydx - 0$. Ans: $y = c(1 + x)$ $\square$

Example 3.5. Solve[3]

$$\frac{dy}{dx} = y^2 - 4 \tag{3.24}$$

Ans:

$$y = 2\frac{1 + Ce^{4x}}{1 - Ce^{4x}} \text{ or } y = \pm 2 \quad \square \tag{3.25}$$

[2] Zill Example 2.2.1

[3] Zill Example 2.2.3

Example 3.6. Solve[4]

$$(e^{2y} - y)\cos x \, \frac{dy}{dx} = e^y \sin 2x, \;\; y(0) = 0 \tag{3.26}$$

Ans:

$$e^y + ye^{-y} + e^{-y} = 4 - 2\cos x \quad \square \tag{3.27}$$

Example 3.7. Solve[5]

$$\frac{dy}{dx} = e^{-x^2}, \;\; y(3) = 5 \tag{3.28}$$

Ans:

$$y(x) = 5 + \int_3^x e^{-t^2}\, dt \quad \square \tag{3.29}$$

Definition 3.1. The **Error Function** $\operatorname{erf}(x)$ is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \tag{3.30}$$

A plot of $\operatorname{erf}(x)$ is given in figure 4.2.

Example 3.8. Rewrite the solution to example 3.7 in terms of $\operatorname{erf}(x)$

$$y(x) = 5 + \frac{\sqrt{\pi}}{2}(\operatorname{erf}(x) - \operatorname{erf}(3)) \quad \square \tag{3.31}$$

Sometimes it is not so easy to tell by looking at an equation if it is separable because it may need to be factored before the variables can be separated. There is a test that we can use that will sometimes help us to disentangle these variables. To derive this test, we will rearrange the general separable equation as follows

$$\frac{dy}{dt} = a(t)b(y) \tag{3.32}$$

$$\frac{dy}{b(y)} = a(t)dt \tag{3.33}$$

$$\frac{dy}{b(y)} - a(t)dt = 0 \tag{3.34}$$

$$N(y)dy + M(t)dt = 0 \tag{3.35}$$

[4]Zill Example 2.2.4
[5]Zill Example 2.2.5

where $M(t) = -a(t)$ and $N(y) = 1/b(y)$ are new names that we give our functions. This gives us the standard form for a separable equation

$$\boxed{M(t)dt + N(y)dy = 0} \tag{3.36}$$

The reason for calling this the standard format will become more clear when we study exact equations. To continue deriving our test for separability we rearrange 3.36 as

$$\frac{dy}{dt} = -\frac{M(t)}{N(y)} \tag{3.37}$$

Recalling the standard form of an ordinary differential equation

$$\frac{dy}{dt} = f(t, y) \tag{3.38}$$

we have

$$f(t, y) = -\frac{M(t)}{N(y)} \tag{3.39}$$

Since M is only a function of t,

$$\frac{\partial M}{\partial t} = M'(t) = \frac{dM}{dt}, \quad \frac{\partial M}{\partial y} = 0 \tag{3.40}$$

and because N is only a function of y,

$$\frac{\partial N}{\partial t} = 0, \quad \frac{\partial N}{\partial y} = N'(y) = \frac{dN}{dy} \tag{3.41}$$

Similarly

$$f_t = \frac{\partial f}{\partial t} = -\frac{M'(t)}{N(y)} \tag{3.42}$$

and

$$f_y = \frac{\partial f}{\partial y} = -\frac{M(t)N'(y)}{N^2(y)} \tag{3.43}$$

The cross-derivative is

$$f_{ty} = \frac{\partial^2 f}{\partial t \partial y} = -\frac{M'(t)N'(y)}{N^2(y)} \tag{3.44}$$

Hence

$$\begin{aligned} f f_{ty} &= \left(-\frac{M(t)}{N(y)}\right)\left(-\frac{M'(t)N'(y)}{N^2(y)}\right) && (3.45)\\ &= \left(-\frac{M'(t)}{N(y)}\right)\left(-\frac{M(t)N'(y)}{N^2(y)}\right) && (3.46)\\ &= f_t f_y && (3.47) \end{aligned}$$

In other words, the equation $y' = f(t, y)$ is separable if and only if

$$\boxed{f(t,y)\frac{\partial^2 f}{\partial t \partial y} = \frac{\partial f}{\partial t}\frac{\partial f}{\partial y}} \tag{3.48}$$

Example 3.9. Determine if

$$\frac{dy}{dt} = 1 + t^2 + y^3 + t^2y^3 \tag{3.49}$$

is separable.

From the right hand side of the differential equation we see that

$$f(t, y) = 1 + t^2 + y^3 + t^2y^3 \tag{3.50}$$

Calculating all the necessary partial derivatives,

$$\frac{\partial f}{\partial t} = 2t + 2ty^3 \tag{3.51}$$

$$\frac{\partial f}{\partial y} = 3y^2 + 3t^2y^2 \tag{3.52}$$

Hence

$$\frac{\partial f}{\partial t}\frac{\partial f}{\partial y} = \left(2t + 2ty^3\right)\left(3y^2 + 3t^2y^2\right) \tag{3.53}$$

$$= 6ty^2 + 6t^3y^2 + 6ty^5 + 6t^3y^5 \tag{3.54}$$

and

$$f(t,y)\frac{\partial^2 f}{\partial t \partial y} = \left(1 + t^2 + y^3 + t^2y^3\right)6ty^2 \tag{3.55}$$

$$= 6ty^2 + 6t^3y^2 + 6ty^5 + 6t^3y^5 \tag{3.56}$$

$$= \frac{\partial f}{\partial t}\frac{\partial f}{\partial y} \tag{3.57}$$

Consequently the differential equation is separable. $\square$

Of course, knowing that the equation is separable does not tell us how to solve it. It does, however, tell us that looking for a factorization of

$$f(t, y) = a(t)b(y) \tag{3.58}$$

is not a waste of time. In the previous example, the correct factorization is

$$\frac{dy}{dt} = (1 + y^3)(1 + t^2) \tag{3.59}$$

Example 3.10. Find a general solution of

$$\frac{dy}{dt} = 1 + t^2 + y^3 + t^2y^3 \tag{3.60}$$

From (3.59) we have

$$\frac{dy}{dt} = (1 + y^3)(1 + t^2) \tag{3.61}$$

hence the equation can be re-written as

$$\int \frac{dy}{1 + y^3} = \int (1 + t^2)dt \tag{3.62}$$

The integral on the left can be solved using the method of partial fractions:

$$\frac{1}{1 + y^3} = \frac{1}{(y + 1)(y^2 - y + 1)} = \frac{A}{1 + y} + \frac{By + C}{y^2 - y + 1} \tag{3.63}$$

Cross-multiplying gives

$$1 = A(y^2 - y + 1) + (By + C)(1 + y) \tag{3.64}$$

Substituting $y = -1$ gives

$$1 = A(1 + 1 + 1) \implies A = \frac{1}{3} \tag{3.65}$$

Sbstituting $y = 0$

$$1 = A(0 - 0 + 1) + (0 + C)(1 + 0) = \frac{1}{3} + C \implies C = \frac{2}{3} \tag{3.66}$$

Using $y = 1$

$$1 = A(1 - 1 + 1) + (B(1) + C)(1 + 1) = \frac{1}{3} + 2B + \frac{4}{3} \tag{3.67}$$

Hence

$$2B = 1 - \frac{5}{3} = -\frac{2}{3} \implies B = -\frac{1}{3} \tag{3.68}$$

Thus

$$
\begin{aligned}
\int \frac{dy}{1+y^3} &= \frac{1}{3}\int \frac{dy}{1+y} + \int \frac{(-1/3)y+2/3}{y^2-y+1}dy && (3.69)\\
&= \frac{1}{3}\ln|1+y| + \int \frac{(-1/3)y+2/3}{y^2-y+1}dy && (3.70)\\
&= \frac{1}{3}\ln|1+y| + \int \frac{(-1/3)y+2/3}{y^2-y+1/4-1/4+1}dy && (3.71)\\
&= \frac{1}{3}\ln|1+y| + \int \frac{(-1/3)y+2/3}{(y-1/2)^2+3/4}dy && (3.72)\\
&= \frac{1}{3}\ln|1+y| - \frac{1}{3}\int \frac{ydy}{(y-1/2)^2+3/4} + \\
&\qquad + \frac{2}{3}\int \frac{dy}{(y-1/2)^2+3/4} && (3.73)\\
&= \frac{1}{3}\ln|1+y| - \frac{1}{6}\ln|y^2-y+1| + \frac{2}{3}\sqrt{\frac{4}{3}}\tan^{-1}\frac{y-1/2}{\sqrt{3/4}} && (3.74)\\
&= \frac{1}{3}\ln|1+y| - \frac{1}{6}\ln|y^2-y+1| + \frac{4}{3\sqrt{3}}\tan^{-1}\frac{2y-1}{\sqrt{3}} && (3.75)
\end{aligned}
$$

Hence the solution of the differential equation is

$$
\frac{1}{3}\ln|1+y| - \frac{1}{6}\ln|y^2-y+1| + \frac{4}{3\sqrt{3}}\tan^{-1}\frac{2y-1}{\sqrt{3}} - t - \frac{1}{3}t^3 = C \quad (3.76)
$$

Lesson 4

Linear Equations

Recall that a function y is linear in a variable x if it describes a straight line, e.g., we write something like

$$y = Ax + B \tag{4.1}$$

to mean that y is linear in x. If we have an algebraic system that depends on t, we might allow A and B to be functions of t, e..g, the equation

$$y = A(t)x + B(t) \tag{4.2}$$

is also linear in x. For example,Recall that a function is linear in a

$$y = t^2 x + 3 \sin t \tag{4.3}$$

is linear in x because for *any fixed value of* t, it has the form

$$y = Ax + B \tag{4.4}$$

Thus to determine the linearity of a function in x, *the nature of the dependence on any other variable does not matter.* The same definition holds for differential equations.

By a **linear differential equation** we mean a differential equation of a single variable, say $y(t)$, whose **derivative depends on itself only linearly**. The nature of the dependence on the time variable does not matter. From our discussion above, something is linear in x if it is written as

$$Ax + B \tag{4.5}$$

where A and B do not depend on x. Hence something is linear in y if it can be written as

$$Ay + B \tag{4.6}$$

if A and B do not depend on y. Thus for a differential equation to be linear in y it must have the form

$$\frac{dy}{dt} = Ay + B \tag{4.7}$$

where neither A nor B depends on y. However, **both A and B are allowed to depend on t.** To emphasize this we write the terms A and B as $A(t)$ and $B(t)$, so that the linear equation becomes

$$\frac{dy}{dt} = A(t)y + B(t) \tag{4.8}$$

For convenience of finding solutions (its not clear now why this is convenient, but trust me) we bring the term in $A(t)y$ to the left hand side of the equation:

$$\frac{dy}{dt} - A(t)y = B(t) \tag{4.9}$$

To be consistent with most textbooks on differential equations we will relabel $A(t) = -p(t)$ and $B(t) = q(t)$, for some function $p(t)$ and $q(t)$, and this gives us

$$\boxed{\frac{dy}{dt} + p(t)y = q(t)} \tag{4.10}$$

which we will refer to as **the standard form** of the linear ODE. Sometimes for convenience we will omit the t (but the dependence will be implied) on the p and q and write the derivative with a prime, as

$$y' + py = q \tag{4.11}$$

There is a general technique that will **always** work to solve a first order linear ODE. We will derive the method constructively in the following paragraph and then give several examples of its use. The idea is look for some function $\mu(t)$ that we can multiply both sides of the equation:

$$\mu \times (y' + py) = \mu \times q \tag{4.12}$$

or

$$\mu y' + \mu py = \mu q \tag{4.13}$$

So far any function μ will work, but not any function will help. We want to find an particular function μ such that the left hand side of the equation becomes

$$\frac{d(\mu y)}{dt} = \mu y' + \mu p y = \mu q \tag{4.14}$$

The reason for looking for this kind of μ is that **if we can find** μ then

$$\frac{d}{dt}(\mu y) = \mu q \tag{4.15}$$

Multiplying both sides by dt and integrating gives

$$\int \frac{d}{dt}(\mu(t)y)dt = \int \mu(t)q(t)dt \tag{4.16}$$

Since the integral of an exact derivative is the function itself,

$$\mu(t)y = \int \mu(t)q(t)dt + C \tag{4.17}$$

hence dividing by μ, we find that **if we can find μ to satisfy equation 4.14 then the general solution of equation 4.10 is**

$$\boxed{y = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)dt + C\right]} \tag{4.18}$$

So now we need to figure out what function $\mu(t)$ will work. From the product rule for derivatives

$$\frac{d}{dt}(\mu y) = \mu y' + \mu' y \tag{4.19}$$

Comparing equations 4.14 and 4.19,

$$\begin{aligned} \mu y' + \mu' y &= \mu y' + \mu p y && (4.20)\\ \mu' y &= \mu p y && (4.21)\\ \mu' &= \mu p && (4.22) \end{aligned}$$

Writing $\mu' = d\mu/dt$ we find that we can rearrange and integrate both sides of the equation:

$$\begin{aligned} \frac{d\mu}{dt} &= \mu p && (4.23)\\ \frac{d\mu}{\mu} &= pdt && (4.24)\\ \int \frac{1}{\mu} d\mu &= \int pdt && (4.25)\\ \ln \mu &= \int pdt + C && (4.26) \end{aligned}$$

Exponentiating both sides of the equation

$$\mu = e^{\int pdt + C} = e^{\int pdt} e^C = C_1 e^{\int pdt} \tag{4.27}$$

where $C_1 = e^C$ and C is *any* constant. Since we want any function μ that will work, we are free to choose our constant arbitrarily, e.g., pick $C = 0$ hence $C_1 = 1$, and we find that

$$\boxed{\mu(t) = e^{\int p(t)dt}} \tag{4.28}$$

has the properties we desire. We say the equation 4.28 is an **integrating factor** for the differential equation 4.10. Since we have already chosen the constant of integrate, *we can safely ignore the constant of integration when integrating* p. To recap, the general solution of $y' + py = q$ is given by equation 4.18 whenever μ is given by 4.28. The particular solution of a given initial value problem involving a linear ODE is then solved by substituting the initial condition into the general solution obtained in this manner.

It is usually easier to memorize the procedure rather than the formula for the solution (equation 4.18):

Method to Solve $y' + p(t)y = q(t)$

1. Computer $\mu = e^{\int p(t)dt}$ and observe that $\mu'(t) = p(t)\mu(t)$.
2. Multiply the ODE through by $\mu(t)$ giving
$$\mu(t)y' + \mu'(t)y = \mu(t)q(t)$$
3. Observe that the left-hand side is precisely $(d/dt)(\mu(t)y)$.
4. Integrate both sides of the equation over t, remembering that $\int (d/dt)(\mu y)dt = \mu(t)y$,
$$\mu(t)y = \int q(t)ydt + C$$
5. Solve for y by dividing through by μ. Don't forget the constant on the right-hand side of the equation.
6. Use the initial condition to find the value of the constant, if this is an initial value problem.

Example 4.1. Solve the differential equation

$$y' + 2ty = t \tag{4.29}$$

This has the form $y' + p(t)y = q(t)$, where $p(t) = 2t$ and $q(t) = t$. An integrating factor is

$$\mu(t) = \exp\left(\int p(t)dt\right) = \exp\left(\int 2tdt\right) = e^{t^2} \tag{4.30}$$

Multiplying equation (4.29) through by the integrating factor $\mu(t)$ gives

$$e^{t^2}(y' + 2ty) = te^{t^2} \tag{4.31}$$

Recall that the left hand side will always end up as the derivative of $y\mu$ after multiplying through by μ; we can also verify this with the product rule:

$$\frac{d}{dt}\left(ye^{t^2}\right) = y'e^{t^2} + 2te^{t^2}y = e^{t^2}(y' + 2ty) \tag{4.32}$$

Comparing the last two equations tells us that

$$\frac{d}{dt}\left(ye^{t^2}\right) = te^{t^2} \tag{4.33}$$

Multiply through both sides by dt and integrate:

$$\int \frac{d}{dt}\left(ye^{t^2}\right)dt = \int te^{t^2}dt \tag{4.34}$$

The left hand side is an exact derivative, hence

$$ye^{t^2} = \int te^{t^2}dt + C \tag{4.35}$$

The right hand side can be solved with the substitution $u = t^2$:

$$\int te^{t^2}dt = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{t^2} + C \tag{4.36}$$

Combining the last two equations,

$$ye^{t^2} = \frac{1}{2}e^{t^2} + C \tag{4.37}$$

Hence we can solve for y,

$$y = \frac{1}{2} + Ce^{-t^2}. \quad \square \tag{4.38}$$

Example 4.2. Solve $ty' - y = t^2 e^{-t}$.

We first need to put this into standard form $y' + p(t)y = q(t)$. If we divide the differential equation on both sides by t then

$$y' - \frac{1}{t}y = te^{-t} \tag{4.39}$$

Hence

$$p(t) = -\frac{1}{t} \tag{4.40}$$

An integrating factor is

$$\mu(t) = \exp\left(\int -\frac{1}{t}dt\right) = \exp(-\ln t) = \exp\left(\ln\frac{1}{t}\right) = \frac{1}{t} \tag{4.41}$$

Multiplying the differential equation (in standard form) through by $\mu(t)$,

$$\frac{1}{t}\left(y - \frac{1}{t}y\right) = \left(\frac{1}{t}\right)(te^{-t}) = e^{-t} \tag{4.42}$$

The left hand side is the exact derivative of μy:

$$\frac{d}{dt}\mu y = \frac{d}{dt}\left(\frac{y}{t}\right) = \frac{ty' - y}{t^2} = \frac{1}{t}\left(y' - \frac{y}{t}\right) \tag{4.43}$$

Hence

$$\frac{d}{dt}\left(\frac{y}{t}\right) = e^{-t} \tag{4.44}$$

Multiplying by dt and integrating,

$$\int \frac{d}{dt}\left(\frac{y}{t}\right)dt = \int e^{-t}dt \tag{4.45}$$

Since the left hand side is an exact derivative,

$$\frac{y}{t} = \int e^{-t}dt + C = -e^{-t} + C \tag{4.46}$$

Solving for y,

$$y = -te^{-t} + Ct \quad \square \tag{4.47}$$

Example 4.3. Solve the initial value problem

$$y' + y = \cos t,\ y(0) = 1 \tag{4.48}$$

Since $p(t) = 1$ (the coefficient of y), an integrating factor is

$$\mu = \exp\left(\int 1 \cdot dt\right) = e^t \tag{4.49}$$

Multiplying the differential equation through by the integrating factor gives

$$\frac{d}{dt}\left(e^t y\right) = e^t(y' + y) = e^t \cos t \tag{4.50}$$

Multiplying by dt and integrating,

$$\int \frac{d}{dt}\left(e^t y\right) dt = \int e^t \cos t dt \tag{4.51}$$

The left hand side is an exact derivative; the right hand side we can find from an integral table or WolframAlpha:

$$ye^t = \int e^t \cos t dt = \frac{1}{2}e^t\left(\sin t + \cos t\right) + C \tag{4.52}$$

The initial condition tells us that $y = 1$ when $t = 0$,

$$(1)e^0 = \frac{1}{2}e^0\left(\sin(0) + \cos(0)\right) + C \tag{4.53}$$

$$1 = \frac{1}{2} + C \tag{4.54}$$

$$C = \frac{1}{2} \tag{4.55}$$

Substituting $C = 1/2$ into equation 4.52 gives

$$ye^t = \frac{1}{2}e^t\left(\sin t + \cos t\right) + \frac{1}{2} \tag{4.56}$$

We can then solve for y by dividing by e^t,

$$y = \frac{1}{2}\left(\sin t + \cos t\right) + \frac{1}{2}e^{-t} \tag{4.57}$$

which is the unique solution to the initial value problem. $\square$

Example 4.4. Solve the initial value problem

$$\left.\begin{aligned} y' - 2y &= e^{7t} \\ y(0) &= 1 \end{aligned}\right\} \tag{4.58}$$

The equation is already given in standard form, with $p(t) = -2$. Hence an integrating factor is

$$\mu(t) = \exp\left(\int_0^t (-2)dt\right) = e^{-2t} \tag{4.59}$$

Multiply the original ODE by the integrating factor $\mu(t)$ gives

$$e^{-2t}(y' - 2y) = (e^{-2t})(e^{7t}) \tag{4.60}$$

Simplifying and recognizing the left hand side as the derivative of μy,

$$\frac{d}{dt}\left(ye^{-2t}\right) = e^{5t} \tag{4.61}$$

Multiply by dt and integrate:

$$\int \frac{d}{dt}\left(ye^{-2t}\right) dt = \int e^{5t}dt \tag{4.62}$$

$$ye^{-2t} = \frac{1}{5}e^{5t} + C \tag{4.63}$$

$$y = \frac{1}{5}e^{7t} + Ce^{2t} \tag{4.64}$$

From the initial condition

$$1 = \frac{1}{5}e^0 + Ce^0 = \frac{1}{5} + C \tag{4.65}$$

Hence $C = 4/5$ and the solution is

$$y = \frac{1}{5}e^{7t} + \frac{4}{5}e^{2t} \quad \square \tag{4.66}$$

Example 4.5. Find the general solutions of

$$ty' + 4y = 4t^2 \tag{4.67}$$

To solve this we first need to put it into standard linear form by dividing by t:

$$y' + \frac{4y}{t} = \frac{4t^2}{t} = 4t \tag{4.68}$$

Since $p(t) = 4/t$ an integrating factor is

$$\mu = \exp\left(\int \frac{4}{t}dt\right) = \exp\left(4\ln t\right) = t^4 \tag{4.69}$$

Multiplying equation 4.68 by μ gives

$$\frac{d}{dt}\left(t^4 y\right) = t^4\left(y' + \frac{4y}{t}\right) = (4t)(t^4) = 4t^5 \tag{4.70}$$

Integrating,

$$\int \frac{d}{dt}(t^4 y)dt = \int 4t^5 dt \tag{4.71}$$

$$t^4 y = \frac{2t^t}{3} + C \tag{4.72}$$

$$y = \frac{2t^2}{3} + \frac{C}{t^4} \quad \square \tag{4.73}$$

The last example is interesting because it demonstrates a differential equation whose solution will behave radically differently depending on the initial value.

Example 4.6. Solve 4.67 with initial conditions of (a) $y(1) = 0$; (b) $y(1) = 1$; and (c) $y(1) = 2/3$.

(a) With $y(1) = 0$ in equation 4.73

$$0 = \frac{(2)(1)}{3} + \frac{C}{1} = \frac{2}{3} + C \tag{4.74}$$

hence $C = -2/3$, and

$$y = \frac{2t^2}{3} - \frac{2}{3t^4} \tag{4.75}$$

(b) With $y(1) = 1$,

$$1 = \frac{(2)(1)}{3} + \frac{C}{1} = \frac{2}{3} + C \tag{4.76}$$

hence $C = 1/3$, and

$$y = \frac{2t^2}{3} + \frac{1}{3t^4} \tag{4.77}$$

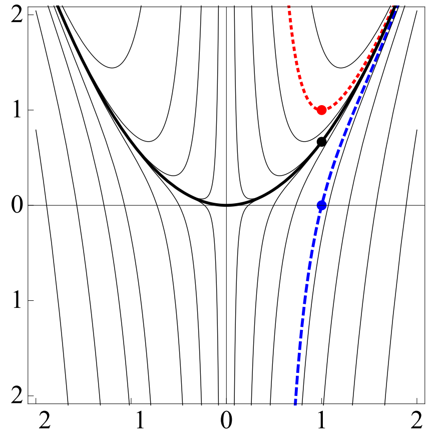
(c) With $y(1) = 2/3$,

$$\frac{2}{3} = \frac{(2)(1)}{3} + \frac{C}{1} = \frac{2}{3} + C \tag{4.78}$$

hence $C = 0$ and

$$y = \frac{2t^2}{3} \tag{4.79}$$

Figure 4.1: Different solutions for example 4.6.



The three solutions are illustrated in the figure 4.1. In cases (a) and (b), the domain of the solution that fits the initial condition excludes $t = 0$; but in case (c), the solution is continuous for all t. Furthermore, $\lim_{t\to 0} y$ is radically different in all three cases. With a little thought we see that

$$\lim_{t\to 0^+} y(t) = \begin{cases} \infty, & \text{if } y(1) > 2/3 \\ 0, & \text{if } y(1) = 2/3 \\ -\infty, & \text{if } y(1) < 2/3 \end{cases} \tag{4.80}$$

As $t \to \infty$ all of the solutions become asymptotic to the curve $y = 2t^2/3$. The side on of the curved asymptote on which the initial conditions falls determines the behavior of the solution for small t. $\square$

What if we were given the initial condition $y(0) = 0$ in the previous example? Clearly the solution $y = 2t^2/3$ satisfies this condition, but if we try to plug the initial condition into the general solution

$$y = \frac{2t^2}{3} + \frac{C}{t^4} \tag{4.81}$$

we have a problem because of the C/t^4 term. One possible approach is to multiply through by the offending t^4 factor:

$$yt^4 = \frac{2}{3}t^6 + C \tag{4.82}$$

Substituting $y = t = 0$ in this immediately yields $C = 0$. This problem is a direct consequence of the fact that we divided our equation through by t^4 previously to get an express solution for $y(t)$ (see the transition from equation 4.72 to equation 4.73): this division is only allowed when $t \neq 0$.

Example 4.7. Solve the initial value problem

$$\left.\begin{aligned} \frac{dy}{dt} - 2ty &= \frac{2}{\sqrt{\pi}} \\ y(0) &= 0 \end{aligned}\right\} \tag{4.83}$$

Since $p(t) = -2t$, an integrating factor is

$$\mu = \exp\left(\int -2t dt\right) = e^{-t^2} \tag{4.84}$$

Following our usual procedure we get

$$\frac{d}{dt}\left(ye^{-t^2}\right) = \frac{2}{\sqrt{\pi}} e^{-t^2} \tag{4.85}$$

If we try to solve the indefinite integral we end up with

$$ye^{-t^2} = \frac{2}{\sqrt{\pi}} \int e^{-t^2} dt + C \tag{4.86}$$

Unfortunately, there is no exact solution for the indefinite integral on the right. Instead we introduce a new concept, of finding a definite integral. We will use as our lower limit of integration the initial conditions, which means $t = 0$; and as our upper limit of integration, some unknown variable u. Then

$$\int_0^u \frac{d}{dt}\left(ye^{-t^2}\right) dt = \int_0^u \frac{2}{\sqrt{\pi}} e^{-t^2} dt \tag{4.87}$$

Then we have

$$\left(ye^{-t^2}\right)_{t=0} - \left(ye^{-t^2}\right)_{t=u} = \frac{2}{\sqrt{\pi}} \int_0^u e^{-t^2} dt \tag{4.88}$$

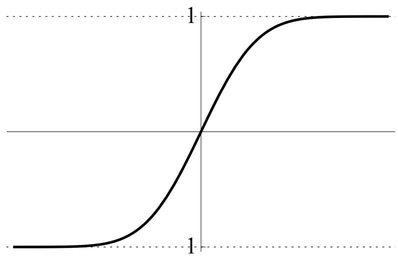
Using the initial condition $y(0) = 0$, the left hand side becomes

$$y(u)e^{-(u)^2} - y(0)e^{-(0)^2} = ye^{-u^2} \tag{4.89}$$

hence

$$ye^{-u^2} = \frac{2}{\sqrt{\pi}} \int_0^u e^{-t^2} dt \tag{4.90}$$

Figure 4.2: The error function, given by equation (4.91).



Note that because of the way we did the integral we have already taken the initial condition into account and hence there is no constant C in the result of our integration.

Now we still do not have a closed formula for the integral on the right but it is a well-defined function, which is called the **error function** and written as

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx \tag{4.91}$$

The error function is a monotonically increasing S-shaped function that passes through the origin and approaches the lines $y = \pm 1$ as $t \to \pm\infty$, as illustrated in figure 4.2. Using equation 4.91 in equation 4.90 gives

$$ye^{-t^2} = \operatorname{erf}(t) \tag{4.92}$$

and solving for y,

$$y = e^{t^2} \operatorname{erf}(t) \tag{4.93}$$

The student should not be troubled by the presence of the error function in the solution since it is a well-known, well-behaved function, just like an exponential or trigonometric function; the only difference is that $\operatorname{erf}(x)$ does not (usually) have a button on your calculator. Most numerical software packages have it built in - for example, in Mathematica it is represented by the function `Erf[x]`. □

This last example gives us another algorithm for solving a linear equation: substitute the initial conditions into the integral in step (4) of the algorithm given on page 28. To see the consequence of this, we return to equation 4.16, but instead take the definite integrals. To have the integration make sense we need to first change the variable of integration to something other than t:

$$\int_{t_0}^{t} \frac{d}{ds}(\mu(s)y(s))ds = \int_{t_0}^{t} \mu(s)q(s)ds \tag{4.94}$$

Evaluating the integral,

$$\mu(t)y(t) - \mu(t_0)y(t_0) = \int_{t_0}^{t} \mu(s)q(s)ds \tag{4.95}$$

Solving for $y(t)$ gives a *general formula for the solution of a linear initial value problem:*

$$\boxed{y(t) = \frac{1}{\mu(t)}\left[\mu(t_0)y(t_0) + \int_{t_0}^{t} \mu(s)q(s)ds\right]} \tag{4.96}$$

Example 4.8. Use equation 4.96 to solve

$$\left.\begin{aligned} t\frac{dy}{dt} + 2y = \frac{4}{t}\sin t \\ y(\pi) = 0 \end{aligned}\right\} \tag{4.97}$$

Rewriting in standard form for a linear differential equation,

$$y' + \frac{2}{t}y = \frac{4}{t^2}\sin t \tag{4.98}$$

Hence $p(t) = 2/t$ and $q(t) = (4/t)^2 \sin t$. An integrating factor is

$$\mu(t) = \exp\left(\int \frac{2}{t}dt\right) = \exp(2\ln t) = t^2 \tag{4.99}$$

Thus equation 4.96 gives us

$$\begin{aligned} y(t) &= \frac{1}{t^2}\left[(\pi^2)(0) + \int_{\pi}^{t} (s^2)\left(\frac{4}{s^2}\sin s\right)ds\right] && (4.100) \\ &= \frac{4}{t^2}\int_{\pi}^{t} \sin s \; ds && (4.101) \\ &= -\frac{4(\cos t - \cos \pi)}{t^2} && (4.102) \\ &= -\frac{4(\cos t + 1)}{t^2} \quad \square && (4.103) \end{aligned}$$

Lesson 5

Bernoulli Equations

The Bernoulli differential equation has the form

$$y' + p(t)y = y^n q(t) \tag{5.1}$$

which is not-quite-linear, because of the factor of y^n on the right-hand side of equation 5.1 would be linear. It does reduce to a linear equation, when either $n = 0$ or $n = 1$. In the first case ($n = 0$), we have

$$y' + p(t)y = q(t) \tag{5.2}$$

which is a general first-order equation. In the second case ($n = 1$)the Bernoulli equation becomes

$$y' + p(t)y = q(t)y \tag{5.3}$$

which can be rearranged to give

$$y' = (q(t) - p(t))y \tag{5.4}$$

This can be solved by multiplying both sides of equation 5.4 by dt/y, and integrating:

$$\int \left(\frac{dy}{dt}\right) dt = \int (q(t) - p(t))dt \tag{5.5}$$

$$y = \int (q(t) - p(t))dt + C \tag{5.6}$$

For any other value of n, Bernoulli equations can be made linear by making the substitution

$$z = y^{1-n} \tag{5.7}$$

Differentiating,

$$\frac{dz}{dt} = (1-n)y^{-n}\frac{dy}{dt} \tag{5.8}$$

Solving for dy/dt,

$$\frac{dy}{dt} = \frac{1}{1-n}y^n\frac{dz}{dt}, \ n \neq 1 \tag{5.9}$$

The restriction to $n \neq 1$ is not a problem because we have already shown how to solve the special case $n = 1$ in equation 5.6.

Substituting equation 5.9 into 5.1 gives

$$\frac{1}{1-n}y^n\frac{dz}{dt} + p(t)y = y^n q(t) \tag{5.10}$$

Dividing through by y^n,

$$\frac{1}{1-n}\frac{dz}{dt} + p(t)y^{1-n} = q(t) \tag{5.11}$$

Substituting from equation 5.7 for z,

$$\frac{1}{1-n}\frac{dz}{dt} + p(t)z = q(t) \tag{5.12}$$

Multiplying both sides of the equation by $1-n$,

$$\frac{dz}{dt} + (1-n)p(t)z = (1-n)q(t) \tag{5.13}$$

which is a linear ODE for z in in standard form.

Rather than writing a formula for the solution it is easier to remember the technique of (a) making a substitution $z = y^{1-n}$; (b) rearranging to get a first-order linear equation in z; (c) solve the ODE for z; and then (d) substitute for y.

Example 5.1. Solve the initial value problem

$$y' + ty = t/y^3 \tag{5.14}$$

$$y(0) = 2 \tag{5.15}$$

This is a Bernoulli equation with n = -3, so we let

$$z = y^{1-n} = y^{1-(-3)} = y^4 \tag{5.16}$$

The initial condition on z is

$$z(0) = y(0)^4 = 2^14 = 16 \tag{5.17}$$

Differentiating equation 5.16

$$\frac{dz}{dt} = 4y^3 \frac{dy}{dt} \tag{5.18}$$

Hence

$$\frac{dy}{dt} = \frac{1}{4y^3}\frac{dz}{dt} \tag{5.19}$$

Substituting equation 5.19 into the original differential equation equation 5.14

$$\frac{1}{4y^3}\frac{dz}{dt} + ty = \frac{t}{y^3} \tag{5.20}$$

Multiplying through by $4y^3$,

$$\frac{dz}{dt} + 4ty^3 = 4t \tag{5.21}$$

Substituting for z from equation 5.16,

$$\frac{dz}{dt} + 4tz = 4t \tag{5.22}$$

This is a first order linear ODE in z with $p(t) = 4t$ and $q(t) = 4t$. An integrating factor is

$$\mu = \exp\left(\int 4tdt\right) = e^{2t^2} \tag{5.23}$$

Multiplying equation 5.22 through by this μ gives

$$\left(\frac{dz}{dt} + 4tz\right) e^{2t^2} = 4te^{2t^2} \tag{5.24}$$

By construction the left hand side must be the exact derivative of $z\mu$; hence

$$\frac{d}{dt}\left(ze^{2t^2}\right) = 4te^{2t^2} \tag{5.25}$$

Multiplying by dt and integrating,

$$\int \frac{d}{dt}\left(ze^{2t^2}\right) dt = \int 4te^{2t^2}\,dt \tag{5.26}$$

Hence

$$ze^{2t^2} = e^{2t^2} + C \tag{5.27}$$

From the initial condition $z(0) = 16$,

$$16e^0 = e^0 + C \implies 16 = 1 + C \implies C = 15 \tag{5.28}$$

Thus

$$z = 1 + 15e^{-2t^2} \tag{5.29}$$

From equation 5.16, $y = z^{1/4}$, hence

$$y = \left(1 + 15e^{-2t^2}\right)^{1/4} \quad \square \tag{5.30}$$

Lesson 6

Exponential Relaxation

One of the most commonly used differential equations used for mathematical modeling has the form

$$\frac{dy}{dt} = \frac{y - C}{\tau} \tag{6.1}$$

where C and τ are constants. This equation is so common that virtually all of the models in section 2.5 of the text, *Modeling with Linear Equations*, take this form, although they are not the only possible linear models.

In the mathematical sciences all variables and constants have some units assigned to them, and in this case the units of C are the same as the units of y, and the units of τ are time (or t). Equation 6.1 is both linear and separable, and we can solve it using either technique. For example, we can separate variables

$$\frac{dy}{y - C} = \frac{dt}{\tau} \tag{6.2}$$

Integrating from (t_0, y_0) to (t, y) (and changing the variables of integration

appropriately first, so that neither t nor y appear in either integrand)[1]

$$\int_{y_0}^{y} \frac{du}{u-C} = \int_{t_0}^{t} \frac{ds}{\tau} \tag{6.3}$$

$$\ln|y-C| - \ln|y_0-C| = \frac{t}{\tau} - \frac{t_0}{\tau} \tag{6.4}$$

$$\ln\left|\frac{y-C}{y_0-C}\right| = \frac{1}{\tau}(t-t_0) \tag{6.5}$$

Exponentiating both sides of the equation

$$\left|\frac{y-C}{y_0-C}\right| = e^{(t-t_0)/\tau} \tag{6.6}$$

hence

$$|y-C| = |y_0-C|e^{(t-t_0)/\tau} \tag{6.7}$$

The absolute value poses a bit of a problem of interpretation. However, the only way that the fraction

$$F = \frac{y-C}{y_0-C} \tag{6.8}$$

can change signs is if it passes through zero. This can only happen if

$$0 = e^{(t-t_0)/\tau} \tag{6.9}$$

which has no solution. So whatever the sign of F, it does not change. At $t = t_0$ we have

$$y - C = y_0 - C \tag{6.10}$$

hence

$$\left|\frac{y-C}{y_0-C}\right| = 1 \tag{6.11}$$

and

$$\frac{y-C}{y_0-C} = 1 \tag{6.12}$$

so by continuity with the initial condition

$$y = C + (y_0 - C)e^{(t-t_0)/\tau} \tag{6.13}$$

It is convenient to consider the two cases $\tau > 0$ and $\tau < 0$ separately.

[1]We must do this because we are using both t and y as endpoints of our integration, and hence must change the name of the symbols we use in the equation, e.g, first we turn (6.2) into $du/(u-C) = ds/\tau$.

Exponential Runaway

First we consider $\tau > 0$. For convenience we will choose $t_0 = 0$. Then

$$y = C + (y_0 - C)e^{t/\tau} \tag{6.14}$$

The solution will either increase exponentially to ∞ or decrease to $-\infty$ depending on the sign of $y_0 - C$.

Example 6.1. Compound interest. If you deposit an amount y_0 in the bank and it accrues interest at a rate r, where r is measured in fraction per year (i.e., $r = 0.03$ means 3% per annum, and it has units of 1/year), and t is measured in years, then the rate at which the current amount on deposit increases is given by

$$\frac{dy}{dt} = ry \tag{6.15}$$

Then we have $C = 0$ and $\tau = 1/r$, so

$$y = y_0 e^{rt} \tag{6.16}$$

So the value increases without bound over time (we don't have a case where $y_0 <$ because that would mean you owe money).

Suppose we also have a fixed salary S per year, and deposit that entirely into our account. Then instead of (6.15), we have

$$\frac{dy}{dt} = ry + S = r(y + S/r) \tag{6.17}$$

In this case we see that $C = -S/r$, the negative ratio of the fixed and interest-based additions to the account. The solution is then

$$y = -\frac{S}{r} + \left(y_0 + \frac{S}{r}\right) e^{rt} \tag{6.18}$$

We still increase exponentially without bounds.

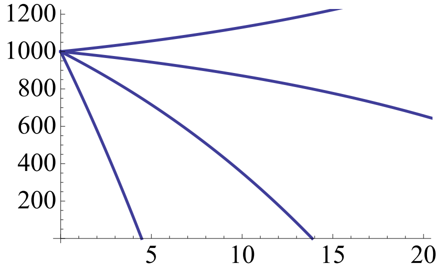
Now consider what happens if instead of depositing a salary we instead withdraw money at a fixed rate of W per year. Since W causes the total amount to decrease, the differential equation becomes

$$\frac{dy}{dt} = ry - W = r(y - W/r) \tag{6.19}$$

Now $C = W/r$. If $W/r < y_0$ then the rate of change will be positive initially and hence positive for all time, and we have

$$y = \frac{W}{r} + \left(y_0 - \frac{W}{r}\right) e^{rt} \tag{6.20}$$

Figure 6.1: Illustration of compound interest at different rates of withdrawal in example 6.1. See text for details.



If instead $W/r > y_0$, the initial rate of change is negative so the amount will always be less than y_0 and your net deposit is decreasing. The term on the right is exponentially increasing, but we can only make withdrawals into the balance is zero. This occurs when the right hand side of the equation is zero.

$$\frac{W}{r} = \left(\frac{W}{r} - y_0\right) e^{rt} \tag{6.21}$$

or at a time given by

$$t = -\frac{1}{r} \ln\left(1 - \frac{y_0 r}{W}\right) \tag{6.22}$$

So if you withdraw money at a rate faster than ry_0 you money will rapidly go away, while if you withdraw at a slower rate, your balance will still increase. Figure 6.1 shows the net balance assuming a starting deposit of \$1000 and a fixed interest rate of 5% per annum, for rates of withdraw (bottom to top) of \$250, \$100, \$60, and \$40 per year. For the example the break-even point occurs when $W = ry_0 = (.05)(1000)$=\$50. □

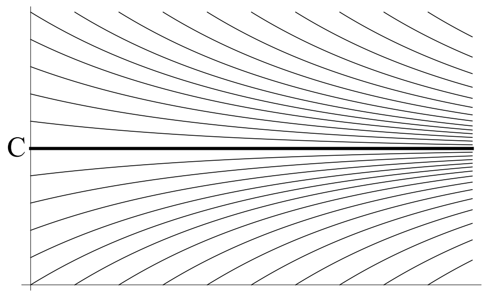
Exponential Relaxation

When $\tau < 0$ we will see that the behavior is quite different - rather then exponential run-away, the solution is pulled to the value of C, whatever the initial value. We will call this phenomenon exponential relaxation.

As before, it is convenient to assume that $t_0 = 0$ and we also define $T = -\tau > 0$ as a positive time constant. Then we have

$$y = C + (y_0 - C)e^{-t/T} \tag{6.23}$$

Figure 6.2: Illustration of the one-parameter family of solutions to $y' = (y - C)/\tau$. As $t \to \infty$, all solutions tend towards C.



If $y_0 - C \neq 0$ then the second term will be nonzero. The exponential factor is always decreasing, so the second term decreases in magnitude with time. Consequently the value of $y \to C$ as $t \to \infty$. To emphasize this we may chose to replace the symbol C with y_∞ (this is common in biological modeling) giving

$$y = y_\infty + (y_0 - y_\infty)e^{-t/\tau} \tag{6.24}$$

Example 6.2. RC-Circuits. An example from engineering is given by RC-circuits (see figure 6.3). The RC refers to a circuit that contains a battery (or other power source); a resistor of resistance R and a capacitor of capacity C. If there is an initial charge q on the capacity, then the charge will decay exponentially if the voltage is sent to zero. RC circuits have the following rules Electric Circuits are governed by the following rules:

1. The voltage drop across a resistor (the difference in voltages between the two ends of the resistor) of resistance R with a current i flowing through it is given by $\Delta V_{resistor} = iR$ (Ohm's Law)

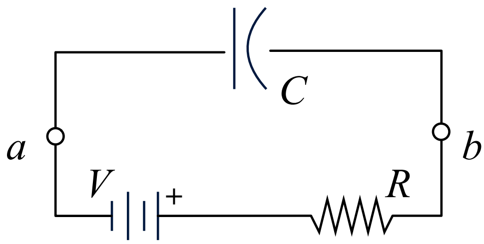
2. Current i represents a flow of charge $i = dq/dt$.

3. If there is a voltage drop across a capacitor of capacitance C, there will be a charge $q = CV$ on the capacitor, hence (by differentiating), there is a current of $i = CdV/dt$.

4. The voltage drop across an inductor of inductance L is $\Delta V_{inductor} = Ldi/dt$.

5. The total voltage drop around a loop must sum to zero.

Figure 6.3: Schematic of RC circuit used in example 6.2.



6. The total current through any node (a point where multiple wires come together) must sum to zero (Kirchoff's Law).

If we were to measure the voltage between points a and b in an RC circuit as illustrated in fig 6.3, these rules tell us first that the capacitor should cause the voltage along the left branch to fluctuate according to

$$i_C = C\frac{dV_{ab}}{dt} \tag{6.25}$$

where i_C is the current through the capacitor. If i_R is the current through the resistor, then the voltage drop through the lower side of the circuit is

$$V_{ab} = V_{batt} + i_R R \tag{6.26}$$

Solving for the current through the resistor,

$$i_R = \frac{V_{ab} - V_{batt}}{R} \tag{6.27}$$

Since the total current around the loop must be zero, $i_C + i_R = 0$,

$$0 = i_R + i_C = \frac{V_{ab} - V_{batt}}{R} + C\frac{dV_{ab}}{dt} \tag{6.28}$$

Dropping the ab subscript,

$$\frac{dV}{dt} = \frac{V_{batt} - V}{RC} \tag{6.29}$$

This is identical to

$$\frac{dV}{dt} = \frac{V_\infty - V}{\tau} \tag{6.30}$$

with

$$V_\infty = V_{batt} \tag{6.31}$$

$$\tau = RC \tag{6.32}$$

Therefore the voltage is given by

$$V = V_{batt} + (V_0 - V_{batt})e^{-t/RC} \tag{6.33}$$

where V_0 is the voltage at $t = 0$. Regardless of the initial voltage, the voltage always tends towards the battery voltage. The current through the circuit is

$$i = C\frac{dV}{dt} = \frac{V_0 - V_{batt}}{R}e^{-t/RC} \tag{6.34}$$

so the current decays exponentially. $\square$

Example 6.3. ***Newton's Law of Heating*** (or Cooling) says that the rate of change of the temperature T of an object (e.g., a potato) is proportional to the difference in temperatures between the object and its environment (e.g., an oven), i.e.,

$$\frac{dT}{dt} = k(T_{oven} - T) \tag{6.35}$$

Suppose that the oven is set to 350 F and a potato at room temperature (70 F) is put in the oven at $t = 0$, with a baking thermometer inserted into the potato. After three minutes you observe that the temperature of the potato is 150. How long will it take the potato to reach a temperature of 350?

The initial value problem we have to solve is

$$\left.\begin{aligned} T' &= k(400 - T) \\ T(0) &= 70 \end{aligned}\right\} \tag{6.36}$$

Dividing and integrating,

$$\int \frac{dT}{400 - T} = \int k dt \tag{6.37}$$

$$-\ln(400 - T) = kt + C \tag{6.38}$$

From the initial condition

$$C = -\ln(400 - 70) = -\ln 330 \tag{6.39}$$

Hence

$$-\ln(400-T) = kt - \ln 330 \tag{6.40}$$

$$\ln\frac{400-T}{330} = -kt \tag{6.41}$$

From the second observation, that $T(3) = 150$,

$$\ln\frac{400-150}{330} = -3k \tag{6.42}$$

we conclude that

$$k = \frac{1}{3}\ln\frac{33}{25} \approx .09 \tag{6.43}$$

From (6.41)

$$\frac{400-T}{330} = e^{-.09t} \tag{6.44}$$

$$T = 400 - 330e^{-.09t} \tag{6.45}$$

The problem asked when the potato will reach a temperature of 350. Returning to (6.41) again,

$$t \approx -\frac{1}{.09}\ln\frac{400-350}{330} \approx 11\ln\frac{33}{5} \approx 20 \text{ minutes} \quad \square \tag{6.46}$$

The motion of an object subjected to external forces $F_1, F_2, \ldots$is given by the solution of the differential equation

Example 6.4. Falling Objects. Suppose an object of constant mass m is dropped from a height h, and is subject to two forces: gravity, and air resistance. The force of gravity can be expressed as

$$F_{gravity} = -mg \tag{6.47}$$

where $g = 9.8\ meters/\sec ond^2$, and the force due to air resistance is

$$F_{drag} = -C_D\frac{dy}{dt} \tag{6.48}$$

where C_D is a known constant, the coefficient of drag; a typical value of $C_D \approx 2.2$. According Newtonian mechanics

$$m\frac{d^2y}{dt^2} = \sum_i F = -mg - C_D\frac{dy}{dt} \tag{6.49}$$

Making the substitution

$$v = \frac{dy}{dt} \tag{6.50}$$

this becomes a first-order ODE in v

$$m\frac{dv}{dt} = -mg - C_D v \tag{6.51}$$

Example 6.5. Suppose we drop a 2010 penny off the top of the empire state building. How long will it take to hit the ground?

The drag coefficient of a penny is approximately $C_D \approx 1$ and its mass is approximately 2.5 grams or 0.0025 kilograms. The height of the observation deck is 1250 feet = 381 meters.

Since the initial velocity is zero, the IVP is

$$\left.\begin{aligned} v' &= -9.8 - v \\ v(0) &= 0 \end{aligned}\right\} \tag{6.52}$$

Rearranging,

$$v' + v = -9.8 \tag{6.53}$$

An integrating factor is e^t so

$$\frac{d}{dt}\left(ve^t\right) = -9.8e^t \tag{6.54}$$

Integrating

$$ve^t = -9.8e^t + C \tag{6.55}$$

Dividing by e^t,

$$v \approx -9.8 + Ce^{-t} \tag{6.56}$$

From the initial condition,

$$0 = -9.8 + C \implies C = 9.8 \tag{6.57}$$

hence

$$v = -9.8 + 9.8e^{-t} \tag{6.58}$$

Now we go back to our substitution of (6.50), we have another first order linear differential equation:

$$\frac{dy}{dt} = -9.8 + 9.8e^{-t} \tag{6.59}$$

Integrating (6.59)

$$y = -9.8t - 9.8e^{-t} + C \tag{6.60}$$

If the object is dropped from a height of 381 meters then

$$y(0) = 381 \tag{6.61}$$

hence

$$381 = -9.8 + C \implies C \approx 391 \tag{6.62}$$

thus

$$y = -9.8t - 9.8e^{-t} + 391 \tag{6.63}$$

The object will hit the ground when $y = 0$, or

$$0 = -9.8t - 9.8e^{-t} + 998 \tag{6.64}$$

This can be solved numerically to give $t \approx 40$ seconds. [2]. $\square$

[2] For example, in Mathematica, we can write `NSolve[0 ==-9.8 t - 9.8 E^-t + 391, t]`

Lesson 7

Autonomous Differential Equations and Population Models

In an **autonomous** differential equation the right hand side does not depend explicitly on t, i.e.,

$$\frac{dy}{dt} = f(y) \tag{7.1}$$

Consequently all autonomous differentiable equations are separable. All of the exponential models discussed in the previous section (e.g., falling objects, cooling, RC-circuits, compound interest) are examples of autonomous differential equations. Many of the basic single-species population models are also autonomous.

Exponential Growth

The exponential growth model was first proposed by Thomas Malthus in 1798.[1] The basic principle is that as members of a population come together, they procreate, to produce more of the species. The rate at which people come together is assumed to be proportional to the the population, and it is assumed that procreation occurs at a fixed rate b. If y is the population, then more babies will be added to the population at a rate by.

[1] In the book *An Essay on the Principle of Population.*

Hence

$$\frac{dy}{dt} = by \tag{7.2}$$

If we assume that people also die at a rate dy then

$$\frac{dy}{dt} = by - dy = (b - d)y = ry \tag{7.3}$$

where $r = b - d$. As we have seen, the solution of this equation is

$$y = y_0 e^{rt} \tag{7.4}$$

Hence if the birth rate exceeds the death rate, then $r = b - d > 0$ and the population will increase without bounds. If the death rate exceeds the birth rate then the population will eventually die if. Only if they are precisely balanced will the population remain fixed.

Logistic Growth

Exponential population growth is certainly seen when resources (e.g., food and water) are readily available and there is no competition; an example is bacterial growth. However, eventually the population will become very large and multiple members of the same population will be competing for the same resources. The members who cannot get the resources will die off. The rate at which members die is thus proportional to the population:

$$d = \alpha y \tag{7.5}$$

so that

$$\frac{dy}{dt} = ry - dy = by - \alpha y^2 = ry\left(1 - \frac{\alpha}{b}y\right) \tag{7.6}$$

for some constant α. It is customary to define

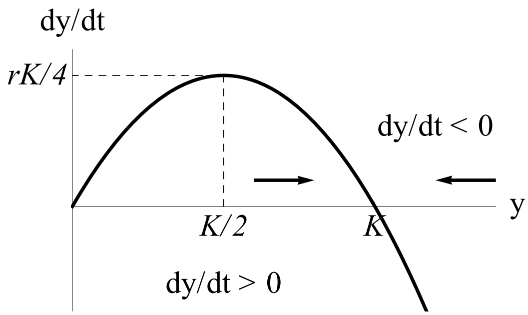
$$K = \frac{r}{\alpha} \tag{7.7}$$

which is called the **carrying capacity** of the population:

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right) \tag{7.8}$$

Equation (7.8) is called the **logistic** growth model, or logistic differential equation. We can analyze this differential equation (without solving it) by looking at the right hand side, which tells us how fast the population increases (or decreases) as a function of population. This is illustrated in figure 7.1

Figure 7.1: A plot of the right-hand side of $y' = f(y)$ for the logistic model in equation 7.8. When $f(y) > 0$, we know that $dy/dt > 0$ and thus y will increase, illustrated by a rightward pointing arrow. Similarly, y decreases when $f(y) < 0$. All arrows point toward the steady state at $y = K$.



When $y < K$, then $dy/dx > 0$, so the population will increase; this is represented by the arrow pointing to the right. When $y > K$, $dy/dt < 0$, so the population will decrease. So no matter what the starting value of y (except for $y = 0$), the arrows always point towards $y = K$. This tells us that the population will approach $y = K$ as time progresses. Thus the carrying capacity tells us the long-term (or steady state) population.

We can solve the logistic model explicitly. In general this is not something we can do in mathematical modeling. hence the frequent use of simple models like exponential or logistic growth. Rewrite (7.8) and separating the variables:

$$K \int \frac{dy}{y(K-y)} = r \int dt \tag{7.9}$$

Using partial fractions,

$$\frac{1}{y(K-y)} = \frac{A}{y} + \frac{B}{K-y} \tag{7.10}$$

Cross multiplying and equating the numerators,

$$1 = A(K-y) + By \tag{7.11}$$

Substituting $y = K$ gives $B = 1/K$; and substituting $y = 0$ gives $A = 1/K$.

Hence

$$\int \frac{dy}{y(K-y)} = \frac{1}{K}\int \frac{dy}{y} + \frac{1}{K}\int \frac{dy}{K-y} \tag{7.12}$$

$$= \frac{1}{K}\left(\ln y - \ln(K-y)\right) \tag{7.13}$$

$$= \frac{1}{K}\ln \frac{y}{K-y} \tag{7.14}$$

Using (7.14) in (7.9)

$$\ln \frac{y}{K-y} = rt + C \tag{7.15}$$

Multiplying through by K and exponentiating,

$$\frac{y}{K-y} = e^{rt+C} = e^{rt}e^{C} \tag{7.16}$$

If we set $y(0) = y_0$ then

$$e^C = \frac{y_0}{K-y_0} \tag{7.17}$$

Hence

$$\frac{y}{K-y} = \frac{y_0}{K-y_0}e^{rt} \tag{7.18}$$

Multiplying by $K-y$,

$$y = K\frac{y_0}{K-y_0}e^{rt} + y\frac{y_0}{K-y_0}e^{rt} \tag{7.19}$$

Bringing the second term to the left and factoring a y,

$$y\left(1 + \frac{y_0}{K-y_0}e^{rt}\right) = K\frac{y_0}{K-y_0}e^{rt} \tag{7.20}$$

Multiplying both sides by $K-y_0$,

$$y\left(K - y_0 + y_0e^{rt}\right) = Ky_0e^{rt} \tag{7.21}$$

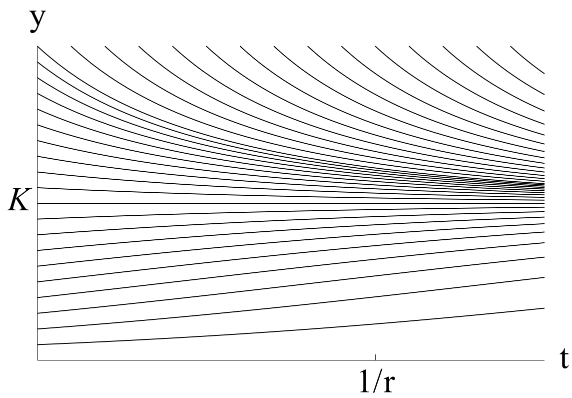
Solving for y,

$$y = \frac{Ky_0e^{rt}}{(K-y_0)+y_0e^{rt}} \tag{7.22}$$

$$= \frac{Ky_0}{y_0 + (K-y_0)e^{-rt}} \tag{7.23}$$

We can see from equation (7.23) that for large t the second term in the denominator approaches zero, hence $y \to K$.

Figure 7.2: Solutions of the logistic growth model, equation (7.23). All nonzero initial populations tend towards the carrying capacity K as $t \to \infty$.



Thresholding Model

If we switch the sign of equation (7.8) it becomes

$$\frac{dy}{dt} = -ry\left(1 - \frac{y}{T}\right) \tag{7.24}$$

where we still assume that $r > 0$ and $T > 0$. The analysis is illustrated below. Instead of all populations approaching T, as it did in the logistic model, all models diverge from T.

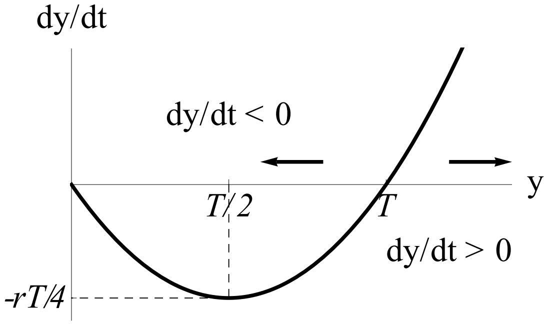
The number T is a **threshold** of the model. At the value $y = T$ the behavior of the solution changes. We expect unlimited growth if $y_0 > T$ and that y will decay towards zero if $y_0 < T$. This type of model describes a species in which there is not sufficient procreation to overcome the death rate unless the initial population is large enough. Otherwise the different members don't run into each other often enough to make new babies. Following the same methods as previously we obtain the solution

$$y = \frac{Ty_0}{y_0 + (T - y_0)e^{rt}} \tag{7.25}$$

We point out the difference between (7.25) and (7.23) - which is the sign of r in the exponential in the denominator.

When $y_0 < T$ then equation (7.25) predicts that $y \to 0$ as $t \to \infty$. It would appear to tell us the same thing for $T < y_0$ but this is misleading.

Figure 7.3: Plot of the right hand side of the thresholding model (7.24). Since $f(y) < 0$ when $y < T$, the population decreases; when $y > T$, the population increases.



Suppose that y_0 is just a little bit larger than T. Then the population because increasing because the right hand side of the equation is positive. As time progresses the second term in the denominator, which is negative, gets larger and larger. This means that the denominator is getting smaller. This causes the population to grow even faster. The denominator becomes zero at $t = t^*$ given by

$$0 = y_0 + (T - y_0)e^{rt^*} \tag{7.26}$$

Solving for t^*

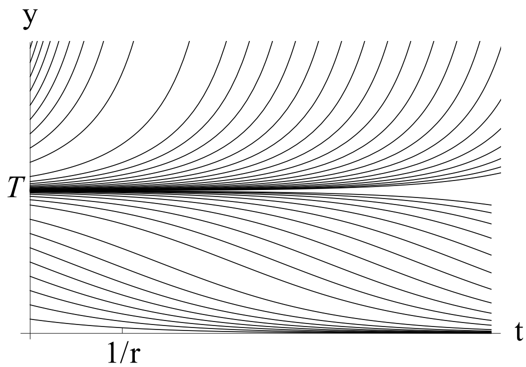
$$t^* = \frac{1}{r} \ln \frac{y_0}{y_0 - T} \tag{7.27}$$

Since $y > T$ the argument of the logarithm is positive, so the solution gives a positive real value for t^*. At this point, the population is predicted to reach ∞; in other words, $t = t^*$ is a vertical asymptote of the solution.

Logistic Growth with a Threshold

The problem with the threshold model is that it blows up. More realistically, when $y_0 > T$, we would expect the rate of growth to eventually decrease as the population gets larger, perhaps eventually reaching some carrying capacity. In other words, we want the population to behave like a thresholding model for low populations, and like a logistic model for larger populations.

Figure 7.4: Solutions of the threshold model given by (7.25). When $y_0 < T$ the population slow decays to zero; when $y_0 > T$ the population increases without bounds, increasing asymptotically to ∞ at a time t^* that depends on y_0 and T (eq. (7.27)).



Suppose we have a logistic model, but let the rate of increase depend on the population:

$$\frac{dy}{dt} = r(y)y\left(1 - \frac{y}{K}\right) \tag{7.28}$$

We want the rate of increase to have a threshold,

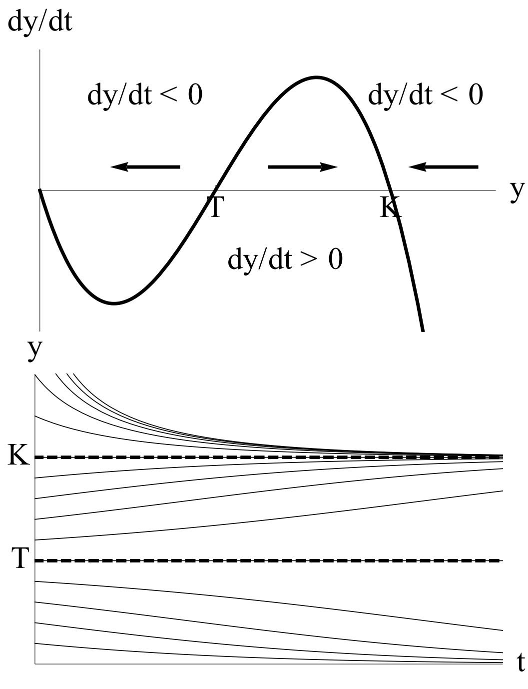
$$r(y) = -r\left(1 - \frac{y}{T}\right) \tag{7.29}$$

combining these two

$$\frac{dy}{dt} = -ry\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right) \tag{7.30}$$

Now we have a growth rate with three zeros at $y = 0$, $y = T$, and $y = K$, where we want $0 < T < K$. If the population exceeds T then it approaches K, while if it is less than T it diminishes to zero.

Figure 7.5: Top: The rate of population change for the logistic model with threshold given by equation (7.30). For $t_0 > T$ all populations tend towards the carrying capacity K; all smaller initial populations decay away towards zero. Bottom: The solutions for different initial conditions.



Lesson 8

Homogeneous Equations

Definition 8.1. An ordinary differential equation is said to be **homogeneous** if it can be written in the form

$$\frac{dy}{dt} = g\left(\frac{y}{t}\right) \tag{8.1}$$

Another way of stating this definition is say that

$$y' = f(t, y) \tag{8.2}$$

is homogeneous if there exists some function g such that

$$y' = g(z) \tag{8.3}$$

where $z = y/t$.

Example 8.1. Show that

$$\frac{dy}{dt} = \frac{2ty}{t^2 - 3y^2} \tag{8.4}$$

is homogeneous.

The equation has the form $y' = f(t, y)$ where

$$f(t,y) = \frac{2ty}{t^2 - 3y^2} \tag{8.5}$$
$$= \frac{2ty}{(t^2)(1 - 3(y/t)^2)} \tag{8.6}$$
$$= \frac{2(y/t)}{1 - 3(y/t)^2} \tag{8.7}$$
$$= \frac{2z}{1 - 3z^2} \tag{8.8}$$

where $z = y/t$. Hence the ODE is homogeneous. □

The following procedure shows that any homogeneous equation can be converted to a separable equation in z by substituting $z = y/t$.

Let $z = y/t$. Then $y = tz$ and thus

$$\frac{dy}{dt} = \frac{d}{dt}(tz) = t\frac{dz}{dt} + z \tag{8.9}$$

Thus if

$$\frac{dy}{dt} = g\left(\frac{y}{t}\right) \tag{8.10}$$

then

$$t\frac{dz}{dt} + z = g(z) \tag{8.11}$$

where $z = y/t$. Bringing the the z to the right-hand side,

$$t\frac{dz}{dt} = g(z) - z \tag{8.12}$$

$$\frac{dz}{dt} = \frac{g(z) - z}{t} \tag{8.13}$$

$$\frac{dz}{g(z) - z} = \frac{dt}{t} \tag{8.14}$$

which is a separable equation in z.

Example 8.2. Find the one-parameter family of solutions to

$$y' = \frac{y^2 + 2ty}{t^2} \tag{8.15}$$

Since

$$y' = \frac{y^2 + 2ty}{t^2} = \frac{y^2}{t^2} + \frac{2ty}{t^2} = \left(\frac{y}{t}\right)^2 + 2\frac{y}{t} = z^2 + 2z \tag{8.16}$$

where $z = y/t$, the differential equation is homogeneous. Hence by equation (8.12) it is equivalent to the following equation in z:

$$t\frac{dz}{dt} + z = z^2 + 2z \tag{8.17}$$

Rearranging and separating variables,

$$t\frac{dz}{dt} = z^2 + z \tag{8.18}$$

$$\frac{dt}{t} = \frac{dz}{z^2 + z} = \frac{dz}{z(1+z)} = \left(\frac{1}{z} - \frac{1}{1+z}\right) dz \tag{8.19}$$

Integrating,

$$\int \frac{dt}{t} = \int \frac{dz}{z} - \int \frac{dz}{1+z} \tag{8.20}$$

$$\ln|t| = \ln|z| - \ln|1+z| + C = \ln\left|\frac{z}{1+z}\right| + C \tag{8.21}$$

Exponentiating,

$$t = \frac{C'z}{1+z} \tag{8.22}$$

for some new constant C'. Dropping the prime on the C and rearranging to solve for z,

$$(1+z)t = Cz \tag{8.23}$$

$$t = Cz - zt = z(C - t) \tag{8.24}$$

$$z = \frac{t}{C - t} \tag{8.25}$$

Since we started the whole process by substituting $z = y/t$ then

$$y = zt = \frac{t^2}{C - t} \tag{8.26}$$

as the general solution of the original differential equation. □

Example 8.3. Solve the initial value problem

$$\left.\begin{aligned} \frac{dy}{dt} &= \frac{y}{t} - 1 \\ y(1) &= 2 \end{aligned}\right\} \tag{8.27}$$

Letting $z = y/t$ the differential equation becomes

$$\frac{dy}{dt} = z - 1 \tag{8.28}$$

Substituting $y' = (zt)' = tz' + z$ gives

$$t\frac{dz}{dt} + z = z - 1 \tag{8.29}$$

Canceling the z's on both sides of the equation

$$\frac{dz}{dt} = -\frac{1}{t} \tag{8.30}$$

$$\int dz = -\int \frac{dt}{t} \tag{8.31}$$

Integrating,

$$z = -\ln t + C \tag{8.32}$$

Since $z = y/t$ then $y = tz$, hence

$$y = t(C - \ln t) \tag{8.33}$$

From the initial condition $y(1) = 2$,

$$2 = (1)(C - \ln 1) = C \tag{8.34}$$

Hence

$$y = t(2 - \ln t) \quad \square \tag{8.35}$$

Lesson 9

Exact Equations

We can re-write any differential equation

$$\frac{dy}{dt} = f(t, y) \tag{9.1}$$

into the form

$$M(t, y)dt + N(t, y)dy = 0 \tag{9.2}$$

Usually there will be many different ways to do this.

Example 9.1. Convert the differential equation $y' = 3t + y$ into the form of equation 9.2.

First we write the equation as

$$\frac{dy}{dt} = 3t + y \tag{9.3}$$

Multiply across by dt:

$$dy = (3t + y)dt \tag{9.4}$$

Now bring all the terms to the left,

$$-(3t + y)dt + dy = 0 \tag{9.5}$$

Here we have

$$M(t, y) = -3t + y \tag{9.6}$$

$$N(t, y) = 1 \tag{9.7}$$

This is not the only way we could have solved the problem. We could have stated by dividing by the right-hand side,

$$\frac{dy}{3t+y} = dt \tag{9.8}$$

To give

$$-\,dt + \frac{1}{3t+y}dy = 0 \tag{9.9}$$

which is also in the desired form, but with

$$M(t,y) = -1 \tag{9.10}$$

$$N(t,y) = \frac{1}{3t+y} \tag{9.11}$$

In general there are infinitely many ways to do this conversion. □

Now recall from calculus that the derivative of a function $\phi(t,y)$ with respect to a third variable, say u, is given by

$$\frac{d}{du}\phi(t,y) = \frac{\partial\phi}{\partial t}\frac{dt}{du} + \frac{\partial\phi}{\partial y}\frac{dy}{du} \tag{9.12}$$

and that the exact differential of $\phi(t,y)$ is obtained from (9.12) by multiplication by du:

$$d\phi(t,y) = \frac{\partial\phi}{\partial t}dt + \frac{\partial\phi}{\partial y}dy \tag{9.13}$$

We can integrate the left hand side:

$$\int d\phi = \phi + C \tag{9.14}$$

If, however, the right hand side is equal to zero, so that $d\phi = 0$, then

$$0 = \int d\phi = \phi + C \implies \phi = C' \tag{9.15}$$

where $C' = -C$ is still a constant.

Let us summarize:

$$\boxed{d\phi = 0 \implies \phi = C} \tag{9.16}$$

for some constant C.

If $d\phi = 0$ then the right hand side of (9.12) is also zero, hence

$$\frac{\partial\phi}{\partial t}dt + \frac{\partial\phi}{\partial y}dy = 0 \implies \phi = C \tag{9.17}$$

Now compare equation (9.2) with (9.17). The earlier equation says that

$$M(t,y)dt + N(t,y)dy = 0 \tag{9.18}$$

This means that if there is some function ϕ such that

$$M = \frac{\partial \phi}{\partial t} \text{ and } N = \frac{\partial \phi}{\partial y} \tag{9.19}$$

Then $\phi = C$ is a solution of the differential equation.

So how do we know when equation (9.19) holds? The answer comes from taking the cross derivatives:

$$\frac{\partial M}{\partial y} = \frac{\partial^2 \phi}{\partial y \partial t} = \frac{\partial^2 \phi}{\partial t \partial y} = \frac{\partial N}{\partial t} \tag{9.20}$$

The second equality follows because the order of partial differentiation can be reversed.

Theorem 9.1. If

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \tag{9.21}$$

then the differential equation

$$M(t,y)dt + N(t,y)dy = 0 \tag{9.22}$$

is called an **exact differential equation** and the solution is given by some function

$$\phi(t,y) = C \tag{9.23}$$

where

$$M = \frac{\partial \phi}{\partial t} \text{ and } N = \frac{\partial \phi}{\partial y} \tag{9.24}$$

We illustrate the method of solution with the following example.

Example 9.2. Solve

$$(2t + y^2)dt + 2tydy = 0 \tag{9.25}$$

This is in the form $Mdt + Ndy = 0$ where

$$M(t,y) = 2t + y^2 \text{ and } N(t,y) = 2ty \tag{9.26}$$

First we check to make sure the equation is exact:

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(2t + y^2) = 2y \tag{9.27}$$

and
$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}(2ty) = 2y \tag{9.28}$$
Since
$$M_y = N_t \tag{9.29}$$
the equation is exact, and therefore the solution is
$$\phi = C \tag{9.30}$$
where
$$\frac{\partial \phi}{\partial t} = M = 2t + y^2 \tag{9.31}$$
and
$$\frac{\partial \phi}{\partial y} = N = 2ty \tag{9.32}$$
To solve the original ODE (9.25) we need to solve the two partial differential equations (9.31) and (9.32). Fortunately we can illustrate a procedure to do this that does not require any knowledge of the methods of partial differential equations, and this method will always work.

Method 1. Start with equation (9.31)
$$\frac{\partial \phi}{\partial t} = 2t + y^2 \tag{9.33}$$
Integrate both sides over t (because the derivative is with respect to t), treating y as a constant in the integration. If we do this, then the constant of integration may depend on y:
$$\int \frac{\partial \phi}{\partial t} dt = \int (2t + y^2) dt \tag{9.34}$$
$$\phi = t^2 + y^2 t + g(y) \tag{9.35}$$
where $g(y)$ is the constant of integration that depends on y. Now substitute (9.35) into (9.32)
$$\frac{\partial}{\partial y}(t^2 + y^2 t + g(y)) = 2ty \tag{9.36}$$
Evaluating the partial derivatives,
$$2yt + \frac{\partial g(y)}{\partial y} = 2ty \tag{9.37}$$
Since $g(y)$ only depends on y, then the partial derivative is the same as $g'(y)$:
$$2yt + \frac{dg}{dy} = 2ty \tag{9.38}$$

Hence

$$\frac{dg}{dy} = 0 \implies g = C' \tag{9.39}$$

for some constant C'. Substituting (9.39) into (9.35) gives

$$\phi = t^2 + y^2 t + C' \tag{9.40}$$

But since $\phi = C$ is a solution, we conclude that

$$t^2 + y^2 t = C'' \tag{9.41}$$

is a solution for some constant C''.

Method 2 Start with equation (9.32) and integrate over y first, treating t as a constant, and treating the constant of integration as a function $h(t)$:

$$\frac{\partial \phi}{\partial y} = 2ty \tag{9.42}$$

$$\int \frac{\partial \phi}{\partial y} dy = \int 2ty dy \tag{9.43}$$

$$\phi = ty^2 + h(t) \tag{9.44}$$

Differentiate with respect to t and use equation (9.31)

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial t}(ty^2 + h(t)) = y^2 + \frac{dh}{dt} = 2t + y^2 \tag{9.45}$$

where the last equality follows from equation (9.31). Thus

$$\frac{dh}{dt} = 2t \implies h = t^2 \tag{9.46}$$

We can ignore the constant of integration because we will pick it up at the end. From equation (9.44),

$$\phi = ty^2 + h(t) = ty^2 + t^2 \tag{9.47}$$

Since $\phi = C$ is the solution of (9.25), we obtain

$$ty^2 + t^2 = C \tag{9.48}$$

is the solution. □

Now we can derive the method in general. Suppose that

$$M(t, y)dt + N(t, y)dy = 0 \tag{9.49}$$

where

$$M_y(t,y) = N_t(t,y) \tag{9.50}$$

Then we conclude that

$$\phi(t,y) = C \tag{9.51}$$

is a solution where

$$\phi_t(t,y) = M(t,y) \text{ and } \phi_y(t,y) = N(t,y) \tag{9.52}$$

We start with the first equation in (9.52), multiply by dt and integrate:

$$\phi = \int \phi_t(t,y)dt = \int M(t,y)dt + g(y) \tag{9.53}$$

Differentiate with respect to y:

$$\phi_y = \frac{\partial}{\partial y}\int M(t,y)dt + g'(y) = \int M_y(t,y)dt + g'(y) \tag{9.54}$$

Hence, using the second equation in (9.52)

$$g'(y) = \phi_y(t,y) - \int M_y(t,y)dt \tag{9.55}$$

$$= N(t,y) - \int M_y(t,y)dt \tag{9.56}$$

Now multiply by dy and integrate:

$$g(y) = \int g'(y)dy = \int \left(N(t,y) - \int M_y(t,y)dt\right) dy \tag{9.57}$$

From equation (9.53)

$$\phi(t,y) = \int M(t,y)dt + \int \left(N(t,y) - \int M_y(t,y)dt\right) dy \tag{9.58}$$

Alternatively, we can start with the second of equations (9.52), multiply by dy (because the derivative is with respect to y), and integrate:

$$\phi = \int \phi_y(t,y)dy = \int N(t,y)dy + h(t) \tag{9.59}$$

where the constant of integration depends, possibly, on t, because we only integrated over y. Differentiate with respect to t:

$$\phi_t(t,y) = \frac{\partial}{\partial t}\int N(t,y)dy + h'(t) = \int N_t(t,y)dy + h'(t) \tag{9.60}$$

From the first of equations (9.52),

$$M(t, y) = \int N_t(t, y)dy + h'(t) \tag{9.61}$$

hence

$$h'(t) = M(t, y) - \int N_t(t, y)dy \tag{9.62}$$

Multiply by dt and integrate,

$$h(t) = \int h'(t)dt = \int \left(M(t, y) - \int N_t(t, y)dy\right) dt \tag{9.63}$$

From (9.59)

$$\phi(t, y) = \int N(t, y)dy + \int \left(M(t, y) - \int N_t(t, y)dy\right) dt \tag{9.64}$$

The following theorem summarizes the derivation.

Theorem 9.2. If $M(t, y)dt + N(t, y)dy = 0$ is exact, i.e., if

$$\frac{\partial M(t, y)}{\partial y} = \frac{\partial N(t, y)}{\partial t} \tag{9.65}$$

The $\phi(t, y) = C$ is a solution of the ODE, where either of the following formulas can be used for ϕ:

$$\phi(t, y) = \int M(t, y)dt + \int \left(N(t, y) - \int M_y(t, y)dt\right) dy \tag{9.66}$$

$$\phi(t, y) = \int N(t, y)dy + \int \left(M(t, y) - \int N_t(t, y)dy\right) dt \tag{9.67}$$

In practice, however, it is generally easier to repeat the derivation rather than memorizing the formula.

Example 9.3. Solve

$$(t + 2y)dt + (2t - y)dy = 0 \tag{9.68}$$

This has the form $M(t, y)dt + N(t, y)dy$ where

$$M(t, y) = t + 2y \tag{9.69}$$

$$N(t, y) = 2t - y \tag{9.70}$$

Checking that it is exact,

$$M_y(t, y) = 2 = N_t(t, y) \tag{9.71}$$

Hence (9.68) is exact and the solution is given by $\phi(t,y)=C$ where

$$\frac{\partial \phi}{\partial t} = M(x,y) = t+2y \tag{9.72}$$

$$\frac{\partial \phi}{\partial y} = N(t,y) = 2t-y \tag{9.73}$$

Integrating (9.73) over t,

$$\phi(t,y) = \int \frac{\partial \phi(t,y)}{\partial t} dt = \int (t+2y)dt = \frac{1}{2}t^2 + 2ty + h(y) \tag{9.74}$$

because the constant of integration may be a function of y. Taking the partial with respect to y and setting the result equal to (9.73)

$$\frac{\partial \phi}{\partial y} = 2t + h'(y) = 2t - y \tag{9.75}$$

Thus

$$h'(y) = -y \implies h(y) = -\frac{y^2}{2} \tag{9.76}$$

we conclude that $h'(y) = -y$ or $h(y) = -y^2/2$. From equation (9.74)

$$\phi(t,y) = \frac{1}{2}t^2 + 2ty - \frac{1}{2}y^2 \tag{9.77}$$

Therefore the solution is

$$\frac{1}{2}t^2 + 2ty - \frac{1}{2}y^2 = C \tag{9.78}$$

for any constant C. □

Example 9.4. Find the one-parameter family of solutions to

$$y' = -\frac{y\cos t + 2te^y}{\sin t + t^2 e^y + 2} \tag{9.79}$$

We can rewrite the equation (9.79) as

$$(y\cos t + 2te^y)dt + (\sin t + t^2 e^y + 2)dy = 0 \tag{9.80}$$

which has the form $M(t,y)dt + N(t,y)dy = 0$ where

$$M(t,y) = y\cos t + 2te^y \tag{9.81}$$

$$N(t,y) = \sin t + t^2 e^y + 2 \tag{9.82}$$

Differentiating equations (9.81) and (9.82) gives

$$\frac{\partial M(t,y)}{\partial y} = \cos t + 2te^y = \frac{\partial N(t,y)}{\partial t} \tag{9.83}$$

and consequently equation (9.79) is exact. Hence the solution is

$$\phi(t,y) = C \tag{9.84}$$

where

$$\frac{\partial \phi}{\partial t} = M(t,y) = y\cos t + 2te^y \tag{9.85}$$

$$\frac{\partial \phi}{\partial y} = N(t,y) = \sin t + t^2e^y + 2 \tag{9.86}$$

Integrating equation (9.85) over t

$$\phi(t,y) = \int \frac{\partial \phi(t,y)}{\partial t} dt = \int (y\cos t + 2te^y)\, dt = y\sin t + t^2e^y + h(y) \tag{9.87}$$

where h is an unknown function of y.

Differentiating (9.87) respect to y and setting the result equal to (9.86) gives

$$\frac{\partial \phi}{\partial y} = \sin t + t^2e^y + h'(y) = \sin t + t^2e^y + 2 \tag{9.88}$$

Thus

$$h'(y) = 2 \implies h(y) = 2y \tag{9.89}$$

Using this back in equation (9.87)

$$\phi(t,y) = y\sin t + t^2e^y + 2y \tag{9.90}$$

Hence the required family of solutiosn is

$$y\sin t + t^2e^y + 2y = C \tag{9.91}$$

for any value of the constant C. □

Example 9.5. Solve the differential equation

$$y' = \frac{at - by}{bt + cy} \tag{9.92}$$

where a, b, c, and d are arbitrary constants.

Rearranging the ODE,

$$(by - at)dt + (bt + cy)dy = 0 \tag{9.93}$$

which is of the form $M(t, y)dt + N(t, y)dy = 0$ with

$$M(t, y) = by - at \tag{9.94}$$
$$N(t, y) = bt + cy \tag{9.95}$$

Since

$$M_y = b = N_t \tag{9.96}$$

equation (9.93) is exact. Therefore the solution is $\phi(t, y) = K$, where

$$\frac{\partial \phi(t, y)}{\partial t} = M(t, y) = by - at \tag{9.97}$$
$$\frac{\partial \phi(t, y)}{\partial y} = N(t, y) = bt + cy \tag{9.98}$$

and K is any arbitrary constant (we don't use C because it is confusing with c already in the problem). Integrating equation in (9.97) over t gives

$$\phi(t, y) = \int (by - at)dt = byt - \frac{a}{2}t^2 + h(y) \tag{9.99}$$

Differentiating with respect to y and setting the result equal to the (9.98),

$$bt + h'(y) = bt + cy \tag{9.100}$$

$$h'(y) = cy \implies h(y) = \frac{c}{2}y^2 \tag{9.101}$$

From equation (9.99)

$$\phi(t, y) = \int (by - at)dt = byt - \frac{a}{2}t^2 + \frac{c}{2}y^2 \tag{9.102}$$

Therefore the solution of (9.92) is

$$\int (by - at)dt = byt - \frac{a}{2}t^2 + \frac{c}{2}y^2 = K \tag{9.103}$$

for any value of the constant K. □

Example 9.6. Solve the initial value problem

$$\left.\begin{aligned} (2ye^{2t} + 2t\cos y)dt + (e^{2t} - t^2 \sin y)dy = 0 \\ y(0) = 1 \end{aligned}\right\} \tag{9.104}$$

This has the form $Mdt + Ndy = 0$ with

$$M(t, y) = 2ye^{2t} + 2t\cos y \tag{9.105}$$

$$N(t, y) = e^{2t} - t^2 \sin y \tag{9.106}$$

Since

$$M_y = 2e^{2t} - 2\sin y = N_t \tag{9.107}$$

we conclude that (9.104) is exact. Therefor the solution of the differential is equation $\phi(t, y) = C$ for some constant C, where

$$\frac{\partial \phi}{\partial t} = M(t, y) = 2ye^{2t} + 2t\cos y \tag{9.108}$$

$$\frac{\partial \phi}{\partial y} = N(t, y) = e^{2t} - t^2 \sin y \tag{9.109}$$

Integrating (9.108) over t

$$\phi(t, y) = \int (2ye^{2t} + 2t\cos y)dt + h(y) = ye^{2t} + t^2\cos y + h(y) \tag{9.110}$$

Taking the partial derivative with respect to y gives

$$\frac{\partial \phi}{\partial y} = e^{2t} - t^2 \sin y + h'(y) \tag{9.111}$$

Comparison of equations (9.111) and (9.109) gives $h'(y) = 0$; hence $h(y)$ is constant. Therefore

$$\phi(t, y) = ye^{2t} + t^2\cos y \tag{9.112}$$

and the general solution is

$$ye^{2t} + t^2\cos y = C \tag{9.113}$$

From the initial condition $y(0) = 1$,

$$(1)(e^0) + (0^2)\cos 1 = C \implies C = 1 \tag{9.114}$$

and therefore the solution of the initial value problem is

$$ye^{2t} + t^2\cos y = 1. \quad \square \tag{9.115}$$

Example 9.7. Solve the initial value problem

$$\left.\begin{aligned}(y/t+\cos t)dt+(e^y+\ln t)dy=0\\ y(\pi)=0\end{aligned}\right\} \tag{9.116}$$

The ODE has the form $Mdt + Ndy = 0$ with

$$M(t,y) = y/t + \cos t \tag{9.117}$$

$$N(t,y) = e^y + \ln t \tag{9.118}$$

Checking the partial derivatives,

$$M_y = \frac{1}{t} = N_t \tag{9.119}$$

hence the differential equation is exact. Thus the solution is $\phi(t,y) = C$ where

$$\frac{\partial \phi}{\partial t} = M(t,y) = \frac{y}{t} + \cos t \tag{9.120}$$

$$\frac{\partial \phi}{\partial y} = N(t,y) = e^y + \ln t \tag{9.121}$$

Integrating the first of these equations

$$\phi(t,y) = \int \left(\frac{y}{t} + \cos t\right) dt + h(y) = y\ln t + \sin t + h(y) \tag{9.122}$$

Differentiating with respect to y and setting the result equal to (9.121),

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}(y\ln t + \sin t + h(y)) = N(t,y) = e^y + \ln t \tag{9.123}$$

$$\ln t + h'(y) = e^y + \ln t \tag{9.124}$$

$$h'(y) = e^y \implies h(y) = e^y \tag{9.125}$$

From (9.122),

$$\phi(t,y) =) = y\ln t + \sin t + e^y \tag{9.126}$$

The general solution of the differential equation is $\phi(t,y) = C$, i.e.,

$$y\ln t + \sin t + e^y = C \tag{9.127}$$

The initial condition is $y(\pi) = 0$; hence

$$(0)\ln\pi + \sin\pi + e^0 = C \implies C = 1 \tag{9.128}$$

Thus

$$y\ln t + \sin t + e^y = 1. \quad \square \tag{9.129}$$

Lesson 10

Integrating Factors

Definition 10.1. An **integrating factor** for the differential equation

$$M(t,y)dt + N(t,y)dy = 0 \tag{10.1}$$

is a any function such $\mu(t,y)$ such that

$$\mu(t,y)M(t,y)dt + \mu(t,y)N(t,y)dy = 0 \tag{10.2}$$

is exact.

If (10.2) is exact, then by theorem 9.1

$$\frac{\partial}{\partial y}(\mu(t,y)M(t,y)) = \frac{\partial}{\partial t}(\mu(t,y)N(t,y)) \tag{10.3}$$

In this section we will discuss some special cases in which we can solve (10.3) to find an integrating factor $\mu(t,y)$ that satisfies (10.3). First we give an example to that demonstrates how we can use an integrating factor.

Example 10.1. Show that

$$\left(\frac{\sin y}{y} - 2e^{-t}\sin t\right)dt + \left(\frac{\cos y + 2e^{-t}\cos t}{y}\right)dy = 0 \tag{10.4}$$

is not exact, and then show that

$$\mu(t,y) = ye^t \tag{10.5}$$

is an integrating factor for (10.4), and then use the integrating factor to find a general solution of (10.4).

We are first asked to verify that (10.4) is not exact. To do this we must show that $M_y \neq N_x$, so we calculate the partial derivatives.

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\sin y}{y} - 2e^{-t}\sin t\right) = \frac{y\cos t - \sin y}{y^2} \tag{10.6}$$

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\cos y + 2e^{-t}\cos t}{y}\right) = -\frac{2}{y}e^{-t}\left(\sin t + \cos t\right) \tag{10.7}$$

Since $M_y \neq N_t$ we may conclude that equation (10.4) is not exact.

To show that (10.5) is an integrating factor, we multiply the differential equation by $\mu(t, y)$. This gives

$$\left(e^t \sin y - 2y\sin t\right) dt + \left(e^t \cos y + 2\cos t\right) dy = 0 \tag{10.8}$$

which is in the form $M(t, y)dt + N(t, y)dy$ with

$$M(t, y) = e^t \sin y - 2y\sin t \tag{10.9}$$

$$N(t, y) = e^t \cos y + 2\cos t \tag{10.10}$$

The partial derivatives are

$$\frac{\partial M(t, y)}{\partial y} = \frac{\partial}{\partial y}(e^t \sin y - 2y\sin t) \tag{10.11}$$

$$= e^t(\cos y - 2\sin t) \tag{10.12}$$

$$\frac{\partial N(t, y)}{\partial t} = \frac{\partial}{\partial t}(e^t \cos y + 2\cos t) \tag{10.13}$$

$$= e^t(\cos y - 2\sin t) \tag{10.14}$$

Since $M_y = N_t$, we may conclude that (10.8) is exact.

Since (10.8) is exact, its solution is $\phi(t, y) = C$ for some function ϕ that satisifes

$$\frac{\partial \phi}{\partial t} = M(t, y) = e^t \sin y - 2y\sin t \tag{10.15}$$

$$\frac{\partial \phi}{\partial y} = N(t, y) = e^t \cos y + 2\cos t \tag{10.16}$$

From (10.15)

$$
\begin{aligned}
\phi(t,y) &= \int \frac{\partial \phi(t,y)}{\partial t} dt + h(y) && (10.17)\\
&= \int M(t,y)dt + h(y) && (10.18)\\
&= \int (e^t \sin y - 2y \sin t)dt + h(y) && (10.19)\\
&= e^t \sin y + 2y \cos t + h(y) && (10.20)
\end{aligned}
$$

Differentiating with respect to y,

$$
\frac{\partial \phi(t,y)}{\partial y} = e^t \cos y + 2 \cos t + h'(y) \qquad (10.21)
$$

From equation (10.16),

$$
e^t \cos y + 2 \cos t + h'(y) = e^t \cos y + 2 \cos t \qquad (10.22)
$$

$$
h'(y) = 0 \implies h(y) = \text{ constant} \qquad (10.23)
$$

Recall the that solution is $\phi(t,y) = C$ where from (10.20), and using $h(y) = K$,

$$
\phi(t,y) = e^t \sin y + 2 \cos t + K \qquad (10.24)
$$

hence

$$
e^t \sin y + 2y \cos t = C \qquad (10.25)
$$

is the general solution of (10.8). □

There is no general method for solving (10.3) to find an integrating factor; however, sometimes we can find an integrating factor that works under certain simplifying assumptions. We will present five of these possible simplifications here as theorems. Only the first case is actually discussed in Boyce & DiPrima (6th Edition).

Theorem 10.2. Integrating Factors, Case 1. If

$$
P^{(1)}(t,y) = \frac{M_y(t,y) - N_t(t,y)}{N(t,y)} \equiv P^{(1)}(t) \qquad (10.26)
$$

is only a function of t, but does not depend on y, then

$$
\mu^{(1)}(t,y) = \exp\left(\int P^{(1)}(t) dt \right) = \exp\left(\int \frac{M_y(t,y) - N_t(t,y)}{N(t,y)} dt \right) \quad (10.27)
$$

is an integrating factor.

Theorem 10.3. Integrating Factors, Case 2. If

$$P^{(2)}(t,y) = \frac{N_t(t,y) - M_y(t,y)}{M(t,y)} \equiv P^{(2)}(y) \tag{10.28}$$

is only a function of y, but does not depend on t, then

$$\mu^{(2)}(t,y) = \exp\left(\int P^{(2)}(y)dt\right) = \exp\left(\int \frac{N_t(t,y) - M_y(t,y)}{M(t,y)}dt\right) \tag{10.29}$$

is an integrating factor.

Theorem 10.4. Integrating Factors, Case 3. If

$$P^{(3)}(t,y) = \frac{N_t(t,y) - M_y(t,y)}{tM(t,y) - yN(t,y)} \equiv P^{(3)}(z) \tag{10.30}$$

is only a function of the product $z = ty$, but does not depend on either t or y in any other way, then

$$\mu^{(3)}(t,y) = \exp\left(\int P^{(3)}(z)dz\right) = \exp\left(\int \frac{N_t(t,y) - M_y(t,y)}{tM(t,y) - yN(t,y)}dz\right) \tag{10.31}$$

is an integrating factor.

Theorem 10.5. Integrating Factors, Case 4. If

$$P^{(4)}(t,y) = \frac{t^2(N_t(t,y) - M_y(t,y))}{tM(t,y) + yN(t,y)} \equiv P^{(4)}(z) \tag{10.32}$$

is only a function of the quotient $z = y/t$, but does not depend on either t or y in any other way, then

$$\mu^{(4)}(t,y) = \exp\left(\int P^{(4)}(z)dz\right) = \exp\left(\int \frac{t^2(N_t(t,y) - M_y(t,y))}{tM(t,y) + yN(t,y)}dz\right) \tag{10.33}$$

is an integrating factor.

Theorem 10.6. Integrating Factors, Case 5. If

$$P^{(5)}(t,y) = \frac{y^2(M_y(t,y) - N_t(t,y))}{tM(t,y) + yN(t,y)} \equiv P^{(5)}(z) \tag{10.34}$$

is only a function of the quotient $z = t/y$, but does not depend on either t or y in any other way, then

$$\mu^{(5)}(t,y) = \exp\left(\int P^{(5)}(z)dz\right) = \exp\left(\int \frac{y^2(M_y(t,y) - N_t(t,y))}{tM(t,y) + yN(t,y)}dz\right) \tag{10.35}$$

is an integrating factor.

Proof. In each of the five cases we are trying to show that $\mu(t, y)$ is an integrating factor of the differential equation

$$M(t, y)dt + N(t, y)dy = 0 \tag{10.36}$$

We have already shown in the discussion leading to equation (10.3) that $\mu(t, y)$ will be an integrating factor of (10.36) if $\mu(t, y)$ satisfies

$$\frac{\partial}{\partial y}\left(\mu(t, y)M(t, y)\right) = \frac{\partial}{\partial t}\left(\mu(t, y)N(t, y)\right) \tag{10.37}$$

By the product rule for derivatives,

$$\mu(t, y)M_y(t, y) + \mu_y(t, y)M(t, y) = \mu(t, y)N_t(t, y) + \mu_t(t, y)N(t, y) \tag{10.38}$$

To prove each theorem, we need to show that under the given assumptions for that theorem, the formula for $\mu(t, y)$ satisfies equation (10.38).

<u>Case 1</u>. If $\mu(t, y)$ is only a functions of t then (a) $\mu_y(t, y) = 0$ (there is no y in the equation, hence the corresponding partial is zero); and (b) $\mu_t(t, y) = \mu'(t)$ (μ only depends on a single variable, t, so there is no distinction between the partial and regular derivative). Hence equation (10.37) becomes

$$\mu(t)M_y(t, y) = \mu(t)N_t(t, y) + \mu'(t)N(t, y) \tag{10.39}$$

Rearranging,

$$\frac{d}{dt}\mu(t) = \frac{M_y(t, y) - N_t(t, y)}{N(t, y)}\mu(t) \tag{10.40}$$

Separating variables, and integrating,

$$\int \frac{1}{\mu(t)}\frac{d}{dt}\mu(t)dt = \int \frac{M_y(t, y) - N_t(t, y)}{N(t, y)}dt \tag{10.41}$$

Hence

$$\ln \mu(t) = \int P^{(1)}(t)dt \implies \mu(t) = \exp\left(\int P^{(1)}(t)dt\right) \tag{10.42}$$

as required by equation (10.27). $\square$(Case 1)

<u>Case 2.</u> If $\mu(t, y) = \mu(y)$ is only a function of y then (a) $\mu_t(t, y) = 0$ (because μ has no t-dependence); and (b) $\mu_y(t, y) = \mu'(t)$ (because μ is only a function of a single variable y). Hence (10.38) becomes

$$\mu(y)M_y(t, y) + \mu'(y)M(t, y) = \mu(y)N_t(t, y) \tag{10.43}$$

Rearranging,

$$\frac{1}{\mu(y)}\frac{d}{dy}\mu(y) = \frac{N_t(t,y) - M_y(t,y)}{M(t,y)} = P^{(2)}(y) \tag{10.44}$$

Multiply by dy and integrating,

$$\int \frac{1}{\mu(y)}\frac{d}{dy}\mu(y)dy = \int P^{(2)}(y)dy \tag{10.45}$$

Integrating and exponentiating

$$\mu(y) = \exp\left(\int P^{(2)}(y)dy\right) \tag{10.46}$$

as required by equation (10.29). □(Case 2).

Case 3. If $\mu(t,y) = \mu(z)$ is only a function of $z = ty$ then

$$\frac{\partial z}{\partial t} = \frac{\partial(ty)}{\partial t} = y \tag{10.47}$$

$$\frac{\partial z}{\partial y} = \frac{\partial(ty)}{\partial y} = t \tag{10.48}$$

Since $\mu(z)$ is only a function of a single variable z we will denote

$$\mu'(z) = \frac{d\mu}{dz} \tag{10.49}$$

By the chain rule and equations (10.47),(10.48), and (10.49),

$$\frac{\partial \mu}{\partial t} = \frac{d\mu}{dz}\frac{\partial z}{\partial t} = \mu'(z)y \tag{10.50}$$

$$\frac{\partial \mu}{\partial y} = \frac{du}{dz}\frac{\partial z}{\partial y} = \mu'(z)t \tag{10.51}$$

Using these results in (10.38),

$$\mu(z)M_y(t,y) + \mu'(z)tM(t,y) = \mu(z)N_t(t,y) + \mu'(z)yN(t,y) \tag{10.52}$$

$$\mu'(z) \times (tM(t,y) - yN(t,y)) = \mu(z) \times (N_t(t,y) - M_y(t,y)) \tag{10.53}$$

$$\frac{\mu'(z)}{\mu(z)} = \frac{N_t(t,y) - M_y(t,y)}{tM(t,y) - yN(t,y))} = P^{(3)}(z) \tag{10.54}$$

Integrating and exponentiating,

$$\mu(z) = \exp\left(\int P^{(3)}(z)dz\right) \tag{10.55}$$

as required by equation (10.31). □(Case 3)

Case 4. If $\mu(t,y)=\mu(z)$ is only a function of $z=y/t$ then

$$\frac{\partial z}{\partial t}=\frac{\partial}{\partial t}\frac{y}{t}=-\frac{y}{t^2} \tag{10.56}$$

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\frac{y}{t}=\frac{1}{t} \tag{10.57}$$

By thee chain rule

$$\frac{\partial \mu}{\partial t}=\frac{d\mu}{dz}\frac{\partial z}{\partial t}=-\mu'\frac{y}{t^2} \tag{10.58}$$

$$\frac{\partial \mu}{\partial y}=\frac{d\mu}{dz}\frac{\partial z}{\partial y}=\frac{\mu'}{t} \tag{10.59}$$

where $\mu'(z)=du/dz$. Equation (10.38) becomes

$$\mu(z)M_y(t,y)+\frac{\mu'(y)}{t}\times M(t,y)=\mu(y)N_t(t,y)+\left(-\mu'(z)\frac{y}{t^2}\right)\times N(t,y) \tag{10.60}$$

Rearranging and solving for $\mu'(z)$

$$\mu(z)(M_y(t,y)-N_t(t,y))=-\mu'(z)\left(\frac{yN(t,y)}{t^2}+\frac{M(t,y)}{t}\right) \tag{10.61}$$

$$=-\frac{\mu'(z)}{t^2}(yN(t,y)+tM(t,y)) \tag{10.62}$$

$$\frac{\mu'(z)}{\mu(z)}=\frac{t^2(N_t(t,y)-M_y(t,y)}{yN(t,y)+tM(t,y)}=P^{(4)}(z) \tag{10.63}$$

Integrating and exponentiating,

$$\mu(z)=\exp\left(\int P^{(4)}zdz\right) \tag{10.64}$$

as required by equation (10.33). □(Case 4)

Case 5. If $\mu(t,y)=\mu(z)$ is only a function of $z=t/y$ then

$$\frac{\partial z}{\partial t}=\frac{\partial}{\partial t}\frac{t}{y}=\frac{1}{y} \tag{10.65}$$

$$\frac{\partial z}{\partial y}=\frac{\partial}{\partial y}\frac{t}{y}=-\frac{t}{y^2} \tag{10.66}$$

By the chain rule,

$$\frac{\partial \mu}{\partial t} = \frac{d\mu}{dz}\frac{\partial z}{\partial t} = \frac{\mu'}{y} \tag{10.67}$$

$$\frac{\partial \mu}{\partial y} = \frac{d\mu}{dz}\frac{\partial z}{\partial y} = -\frac{\mu' t}{y^2} \tag{10.68}$$

where $\mu'(z) = du/dz$. Substituting this into equation (10.38) gives

$$\mu(z)M_y(t,y) + \left(\frac{-\mu(z)t}{y^2}\right)M(t,y) = \mu(z)N_t(t,y) + \left(\frac{\mu'(z)}{y}\right)N(t,y) \tag{10.69}$$

Rearranging

$$\mu(z)(M_y(t,y) - N_t(t,y)) = \left(\frac{\mu'(z)}{y}\right)N(t,y) + \left(\frac{\mu(z)t}{y^2}\right)M(t,y) \tag{10.70}$$

$$= \frac{\mu'(z)}{y^2}(yN(t,y) + tM(t,y)) \tag{10.71}$$

$$\frac{\mu'(z)}{\mu(z)} = \frac{y^2(M_y(t,y) - N_t(t,y))}{yN(t,y) + tM(t,y)} = P^{(5)}(z) \tag{10.72}$$

Multiplying by dz, integrating, and exponentiating gives

$$\mu(z) = \exp\left(\int P^{(5)}(z)dz\right) \tag{10.73}$$

as required by equation (10.35). □(Case 5)

This completes the proof for all five case. □

Example 10.2. . Solve the differential equation

$$(3ty + y^2) + (t^2 + ty)y' = 0 \tag{10.74}$$

by finding an integrating factor that makes it exact.

This equation has the form $Mdt + Ndy$ where

$$M(t,y) = 3ty + y^2 \tag{10.75}$$

$$N(t,y) = t^2 + ty \tag{10.76}$$

First, check to see if the equation is already exact.

$$M_y = 3t + 2y \tag{10.77}$$

$$N_t = 2t + y \tag{10.78}$$

Since $M_y \neq N_t$, equation (10.74) is not exact.

We proceed to check cases 1 through 5 to see if we can find an integrating factor.

$$P^{(1)}(t,y) = \frac{M_y - N_t}{N} \tag{10.79}$$

$$= \frac{(3t+2y)-(2t+y)}{t^2+ty} \tag{10.80}$$

$$= \frac{t+y}{t^2+ty} = \frac{1}{t} \tag{10.81}$$

This depends only on t, hence we can use case (1). The integrating factor is

$$\mu(t) = \exp\left(\int P(t)dt\right) = \exp\left(\int \frac{1}{t}dt\right) = \exp(\ln t) = t \tag{10.82}$$

Multiplying equation (10.74) by $\mu(t) = t$ gives

$$(3t^2y + y^2t)dt + (t^3 + t^2y)dy = 0 \tag{10.83}$$

Equation (10.83) has

$$M(t,y) = 3t^2y + y^2t \tag{10.84}$$

$$N(t,y) = t^3 + t^2y \tag{10.85}$$

This time, since

$$M_y = 3t^2 + 2yt = N_t \tag{10.86}$$

we have an exact equation.

The solution of (10.83) is $\phi(t,y) = C$, where C is an arbitrary constant, and

$$\frac{\partial\phi}{\partial t} = M(t,y) = 3t^2y + ty^2 \tag{10.87}$$

$$\frac{\partial\phi}{\partial y} = N(t,y) = t^3 + t^2y \tag{10.88}$$

To find $\phi(t,y)$ we begin by integrating (10.87) over t:

$$\phi(t,y) = \int \frac{\partial\phi}{\partial t}dt = \int (3t^2 + ty^2)dt = t^3y + \frac{1}{2}t^2y^2 + h(y) \tag{10.89}$$

Differentiating with respect to y

$$\frac{\partial\phi}{\partial y} = t^3 + t^2y + h'(y) \tag{10.90}$$

Equating the right hand sides of (10.88) and (10.90) gives

$$t^3 + t^2 y + h'(y) = t^3 + t^2 y \tag{10.91}$$

Therefore $h'(y) = 0$ and $h(y)$ is a constant. From (10.89), general solution of the differential equation, which is $\phi = C$, is given by

$$t^3 y + \frac{1}{2} t^2 y^2 = C. \quad \square \tag{10.92}$$

Example 10.3. Solve the initial value problem

$$\left.\begin{aligned} (2y^3 + 2)dt + 3ty^2 dy = 0 \\ y(1) = 4 \end{aligned}\right\} \tag{10.93}$$

This has the form $M(t, y)dt + N(t, y)dy$ with

$$M(t, y) = 2y^3 + 2 \tag{10.94}$$

$$N(t, y) = 3ty^2 \tag{10.95}$$

Since

$$M_y = 6y^2 \tag{10.96}$$

$$N_t = 3y^2 \tag{10.97}$$

To find an integrating factor, we start with

$$P(t, y) = \frac{M_y - N_t}{N} = \frac{6y^2 - 3y^2}{3ty^2} = \frac{3y^2}{3ty^2} = \frac{1}{t} \tag{10.98}$$

This is only a function of t and so an integrating factor is

$$\mu = \exp\left(\int \frac{1}{t} dt\right) = \exp(\ln t) = t \tag{10.99}$$

Multiplying (10.93) by $\mu(t) = t$,

$$(2ty^3 + 2t)dt + 3t^2 y^2 dy = 0 \tag{10.100}$$

which has

$$M(t, y) = 2ty^3 + 2t \tag{10.101}$$

$$N(t, y) = 3t^2 y^2 \tag{10.102}$$

Since

$$M_y(t, y) = 6ty^2 = N_t(t, y) \tag{10.103}$$

the revised equation (10.100) is exact and therefore the solution is $\phi(t, y) = C$ where

$$\frac{\partial\phi}{\partial t} = M(t, y) = 2ty^3 + 2t \tag{10.104}$$

$$\frac{\partial\phi}{\partial y} = N(t, y) = 3t^2y^2 \tag{10.105}$$

Integrating (10.104) with over t,

$$\phi(t, y) = \int \frac{\partial\phi}{\partial t} dt + h(y) \tag{10.106}$$

$$= \int (2ty^3 + 2t)dt + h(y) \tag{10.107}$$

$$= t^2y^3 + t^2 + h(y) \tag{10.108}$$

Differentiating with respect to y,

$$\frac{\partial\phi(t, y)}{\partial y} = 3t^2y + h'(y) \tag{10.109}$$

Equating the right hand sides of (10.109) and (10.105) gives

$$3t^2y^2 + h'(y) = 3t^2y^2 \tag{10.110}$$

Hence $h'(y) = 0$ and $h(y) = C$ for some constant C. From (10.108) the general solution is

$$t^2(y^3 + 1) = C \tag{10.111}$$

Applying the initial condition $y(1) = 4$,

$$(1)^2(4^3 + 1) = C \implies C = 65 \tag{10.112}$$

Therefore the solution of the initial value problem (10.93) is

$$t^2(y^3 + 1) = 65 \quad \square \tag{10.113}$$

Example 10.4. Solve the initial value problem

$$\left.\begin{aligned} ydt + (2t - ye^y)dy = 0 \\ y(0) = 1 \end{aligned}\right\} \tag{10.114}$$

Equation (10.114) has the form $M(t, y)dt + N(t, y)dy = 0$ with $M(t, y) = y$ and $N(t, y) = 2t - ye^y$. Since $M_y = 1 \neq 2 = N_t$ the differential equation is not exact. We first try case 1:

$$P^{(1)}(t, y) = \frac{M_y(t, y) - N_t(t, y)}{N(t, y)} = \frac{1}{ye^y - 2t} \tag{10.115}$$

This depends on both t and y; case 1 requires that $P^{(1)}(t,y)$ only depend on t. Hence case 1 fails. Next we try case 2

$$P^{(2)}(t,y) = \frac{N_t(t,y) - M_y(t,y)}{M(t,y)} = \frac{1}{y} \tag{10.116}$$

Since $P^{(2)}$ is purely a function of y, the conditions for case (2) are satisfied. Hence an integrating factor is

$$\mu(y) = \exp\left(\int (1/y)dy\right) = \exp(\ln y) = y \tag{10.117}$$

Multiplying equation the differential equation through $\mu(y) = y$ gives

$$y^2dt + (2ty - y^2e^y)dy = 0 \tag{10.118}$$

This has

$$M(t,y) = y^2 \tag{10.119}$$

$$N(t,y) = 2ty - y^2e^y \tag{10.120}$$

Since $M_y = 2y = N_t$, equation (10.118) is exact. Thus we know that the solution is $\phi(t,y) = C$ where

$$\frac{\partial \phi}{\partial t} = M(t,y) = y^2 \tag{10.121}$$

$$\frac{\partial \phi}{\partial y} = N(t,y) = 2ty - y^2e^y \tag{10.122}$$

Integrating (10.121) over t,

$$\phi(t,y) = \int \frac{\partial \phi}{\partial t}dt + h(y) = \int y^2dt + h(y) = y^2t + h(y) \tag{10.123}$$

Differentiating with respect to y,

$$\frac{\partial \phi}{\partial y} = 2ty + h'(y) \tag{10.124}$$

Equating the right hand sides of (10.124) and (10.122),

$$2yt + h'(y) = 2ty - y^2e^y \tag{10.125}$$

$$h'(y) = -y^2e^y \tag{10.126}$$

$$h(y) = \int h'(y)dy \tag{10.127}$$

$$= -\int y^2e^ydy \tag{10.128}$$

$$= -e^y(2 - 2y + y^2) \tag{10.129}$$

Substituting (10.129) into (10.123) gives

$$\phi(t,y) = y^2 t - e^y(2 - 2y + y^2) \tag{10.130}$$

The general solution of the ODE is

$$y^2 t + e^y(-2 + 2y - y^2) = C \tag{10.131}$$

The initial condition $y(0) = 1$ gives

$$C = (1^2)(0) + e^1(-2 + (2)(1) - 1^2) = -e \tag{10.132}$$

Hence the solution of the initial value problem is

$$y^2 t + e^y(-2 + 2y - y^2) = -e \quad \square \tag{10.133}$$

Lesson 11

Method of Successive Approximations

The following theorem tells us that any initial value problem has an equivalent integral equation form.

Theorem 11.1. The initial value problem

$$\left.\begin{aligned} y'(t,y) &= f(t,y) \\ y(t_0) &= y_0 \end{aligned}\right\} \tag{11.1}$$

has a solution if and only if the integral equation

$$\phi(t) = y_0 + \int_0^t f(s,\phi(s))ds \tag{11.2}$$

has the same solution.

Proof. This is an "if and only if" theorem, so to prove it requires showing two things: (a) that (11.2) $\implies$ (11.1); and (b) that (11.1) $\implies$ (11.2).

To prove (a), we start by assuming that (11.1) is true; we then need to show that (11.2) follows as a consequence.

If (11.1) is true then it has a solution $y = \phi(t)$ that satisfies

$$\left.\begin{aligned} \frac{d\phi}{dt} &= f(t,\phi(t)) \\ \phi(t_0) &= y_0 \end{aligned}\right\} \tag{11.3}$$

Let us change the variable t to s,

$$\frac{d\phi(s)}{ds} = f(s, \phi(s)) \tag{11.4}$$

If we multiply by ds and integrate from $s = t_0$ to $s = t$,

$$\int_{t_0}^{t} \frac{d\phi(s)}{ds} ds = \int_{t_0}^{t} f(s, \phi(s)) ds \tag{11.5}$$

By the fundamental theorem of calculus, since $\phi(s)$ is an antiderivative of $d\phi(s)/ds$, the left hand side becomes

$$\int_{t_0}^{t} \frac{d\phi(s)}{ds} ds = \phi(t) - \phi(t_0) = \phi(t) - y_0 \tag{11.6}$$

where the second equality follows from the second line of equation (11.3).

Comparing the right-hand sides of equations (11.5) and (11.6) we find that

$$\phi(t) - y_0 = \int_{t_0}^{t} f(s, \phi(s)) ds \tag{11.7}$$

Bringing the y_0 to the right hand side of the equation gives us equation (11.2) which was the equation we needed to derive. This completes the proof of part (a).

To prove part (b) we assume that equation (11.2) is true and need to show that equation (11.1) follows as a direct consequence. If we differentiate both sides of (11.2),

$$\begin{align}
\frac{d}{dt}\phi(t) &= \frac{d}{dt}\left(y_0 + \int_0^t f(s, \phi(s)) ds\right) \tag{11.8}\\
&= \frac{dy_0}{dt} + \frac{d}{dt}\int_0^t f(s, \phi(s)) ds \tag{11.9}\\
&= \phi(t, \phi(t)) \tag{11.10}
\end{align}$$

where the last equation follows from the fundamental theorem of calculus. Changing the name of the variable from ϕ to y in (11.10) gives us $y' = f(t, y)$, which is the first line (11.1).

To prove that the second line of (11.1) follows from (11.2), we substitute $t = t_0$ in (11.2).

$$\phi(t_0) = y_0 + \int_0^{t_0} f(s, \phi(s)) ds = y_0 \tag{11.11}$$

because the integral is zero (the top and bottom limits are identical). Changing the label ϕ to y in equation (11.11) returns the second line of (11.1), thus completing the proof of part (b). $\square$

The Method of Successive Approximations, which is also called **Picard Iteration**, attempts to find a solution to the initial value problem (11.1) by solving the integral equation (11.2). This will work because both equations have the same solution. The problem is that solving the integral equation is no easier than solving the differential equation.

The idea is this: generate the sequence of functions $\phi_0, \phi_1, \phi_2, \dots,$ defined by

$$\phi_0(t) = y_0 \tag{11.12}$$

$$\phi_1(t) = y_0 + \int_{t_0}^{t} f(s, \phi_0(s))ds \tag{11.13}$$

$$\phi_2(t) = y_0 + \int_{t_0}^{t} f(s, \phi_1(s))ds \tag{11.14}$$

$$\vdots$$

$$\phi_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, \phi_n(s))ds \tag{11.15}$$

$$\vdots$$

From the pattern of the sequence of functions, we try to determine

$$\phi(t) = \lim_{n\to\infty} \phi_n(t) \tag{11.16}$$

If this limit exists, then it converges to the solution of the initial value problem.

Theorem 11.2. Suppose that $f(t, y)$ and $\partial f(t, y)/\partial y$ are continuous in some box

$$t_0 - a \leq t \leq t_0 + a \tag{11.17}$$

$$y_0 - b \leq y \leq y_0 + b \tag{11.18}$$

then there is some interval

$$t_0 - a \leq t_0 - h \leq t \leq t_0 + h \leq t_0 + a \tag{11.19}$$

in which the Method of Successive Approximations converges to the unique solution of the initial value problem

$$\left.\begin{aligned} \frac{dy}{dt} &= f(t, y) \\ y(t_0) &= y_0 \end{aligned}\right\} \tag{11.20}$$

The proof of this theorem is quite involved and will be discussed in the sections 12 and 13.

The procedure for using the Method of Successive Approximations is summarized in the following box.[1]

Procedure for Picard Iteration

To solve $y' = f(t, y)$ with initial condition $y(t_0) = y_0$:

1. Construct the first 3 iterations $\phi_0, \phi_1, \phi_2, \phi_3$.
2. Attempt to identify a pattern; if one is not obvious you may need to calculate more ϕ_n.
3. Write a formula for the general $\phi_n(t)$ from the pattern.
4. Prove that when you plug $\phi_n(t)$ into the right hand side of equation (11.15) you get the same formula for ϕ_{n+1} with n replaced by $n + 1$.
5. Prove that $\phi(t) = \lim_{n\to\infty} \phi_n$ converges.
6. Verify that $\phi(t)$ solve the original differential equation and initial condition.

[1]The Method of Successive Approximations is usually referred to as Picard iteration for Charles Emile Picard (1856-1941) who popularized it in a series of textbooks on differential equations and mathematical analysis during the 1890's. These books became standard references for a generation of mathematicians. Picard attributed the method to Hermann Schwartz, who included it in a *Festschrift* honoring Karl Weierstrass' 70'th birthday in 1885. Guisseppe Peano (1887) and Ernst Leonard Lindeloff (1890) also published versions of the method. Since Picard was a faculty member at the Sorbonne when Lindeloff, also at the Sorbonne, published his results, Picard was certainly aware of Lindeloff's work. A few authors, including Boyce and DiPrima, mention a special case published by Joseph Liouville in 1838 but I haven't been able to track down the source, and since I can't read French, I probably won't be able to answer the question of whether this should be called Liouville iteration anytime soon.

Example 11.1. Construct the Picard iterates of the initial value problem

$$\left.\begin{aligned} y' &= -2ty \\ y(0) &= 1 \end{aligned}\right\} \tag{11.21}$$

and determine if they converge to the solution.

In terms of equations (11.12) through (11.15), equation (11.21) has

$$f(t,y) = -2ty \tag{11.22}$$
$$t_0 = 0 \tag{11.23}$$
$$y_0 = 1 \tag{11.24}$$

Hence from (11.12)

$$\phi_0(t) = y_0 = 1 \tag{11.25}$$
$$f(s,\phi_0(s)) = -2s\phi_0(s) = -2s \tag{11.26}$$
$$\phi_1(t) = y_0 + \int_0^t f(s,\phi_0(s))ds \tag{11.27}$$
$$= 1 - 2\int_0^t s ds \tag{11.28}$$
$$= 1 - t^2 \tag{11.29}$$

We then use ϕ_1 to calculate $f(s,\phi_1(s))$ and then ϕ_2:

$$f(s,\phi_1(s)) = -2s\phi_1(s) = -2s(1-s^2) \tag{11.30}$$
$$\phi_2(t) = y_0 + \int_0^t f(s,\phi_1(s))ds \tag{11.31}$$
$$= 1 - 2\int_0^t s(1-s^2)ds \tag{11.32}$$
$$= 1 - t^2 + \frac{1}{2}t^4 \tag{11.33}$$

Continuing as before, use ϕ_2 to calculate $f(s,\phi_2(s))$ and then ϕ_3:

$$f(s,\phi_2(s)) = -2s\phi_2(s) = -2s\left(1 - s^2 + \frac{1}{2}s^4\right) \tag{11.34}$$
$$\phi_3(s) = y_0 + \int_0^t f(s,\phi_2(s))ds \tag{11.35}$$
$$= 1 - 2\int_0^t s\left(1 - s^2 + \frac{1}{2}s^4\right)ds \tag{11.36}$$
$$= 1 - t^2 + \frac{1}{2}t^4 - \frac{1}{6}t^6 \tag{11.37}$$

Continuing as before, use ϕ_3 to calculate $f(s, \phi_3(s))$ and then ϕ_4:

$$f(s, \phi_3(s)) = -2s\phi_3(s) = -2s\left(1 - s^2 + \frac{1}{2}s^4 - \frac{1}{6}s^6\right) \tag{11.38}$$

$$\phi_4(s) = y_0 + \int_0^t f(s, \phi_3(s))ds \tag{11.39}$$

$$= 1 - 2\int_0^t \left(s - s^3 + \frac{1}{2}s^5 - \frac{1}{6}s^7\right) ds \tag{11.40}$$

$$= 1 - t^2 + \frac{t^2}{2} - \frac{t^6}{6} + \frac{t^8}{24} \tag{11.41}$$

That pattern that *appears* to be emerging is that

$$\phi_n(t) = \frac{t^2 \cdot 0}{0!} - \frac{t^{2\cdot 1}}{1!} + \frac{t^{2\cdot 2}}{2!} - \frac{t^{2\cdot 3}}{3!} + \cdots + \frac{(-1)^n t^{2n}}{n!} = \sum_{k=0}^{n} \frac{(-1)^k t^{2k}}{k!} \tag{11.42}$$

The only way to know if (11.42) is the correct pattern is to plug it in and see if it works. First of all, it works for all of the n we've already calculated, namely, $n = 1, 2, 3, 4$. To prove that it works for all n we use the principal of **mathematical induction**: A statement $P(n)$ is true for all n if and only if (a) $P(1)$ is true; and (b) $P(n) \implies P(n-1)$. We've already proven (a). To prove (b), we need to show that

$$\phi_{n+1}(t) = \sum_{k=0}^{n+1} \frac{(-1)^k t^{2k}}{k!} \tag{11.43}$$

logically follows when we plug equation (11.42) into (11.15). The reason for using (11.15) is because it gives the general definition of any Picard iterate ϕ_n in terms of the previous iterate. From (11.15), then

$$\phi_{n+1}(t) = y_0 + \int_{t_0}^t f(s, \phi_n(s))ds \tag{11.44}$$

To evaluate (11.44) we need to know $f(s, \phi_n(s))$, based on the expression for $\phi_n(s)$ in (11.42):

$$f(s, \phi_n(s)) = -2s\phi_n(s) \tag{11.45}$$

$$= -2s \sum_{k=0}^{n} \frac{(-1)^k s^{2k}}{k!} \tag{11.46}$$

$$= -2 \sum_{k=0}^{n} \frac{(-1)^k s^{2k+1}}{k!} \tag{11.47}$$

We can then plug this expression for $f(s, \phi_n(s))$ into (11.44) to see if it gives us the expression for ϕ_{n+1} that we are looking for:

$$\phi_{n+1}(t) = 1 - 2\sum_{k=0}^{n} \frac{(-1)^k}{k!} \int_0^t s^{2k+1} ds \tag{11.48}$$

$$= 1 - 2\sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{t^{2k+2}}{2k+2} \tag{11.49}$$

$$= 1 + \sum_{k=0}^{n} \frac{(-1)^{k+1}}{k!} \frac{t^{2k+2}}{k+1} \tag{11.50}$$

$$= \frac{-1^0 t^(2\dot{0})}{0!} + \sum_{k=0}^{n} \frac{(-1)^{k+1}}{(k+1)!} t^{2(k+1)} \tag{11.51}$$

The trick now is to change the index on the sum. Let $j = k+1$. Then $k = 0 \implies j = 1$ and $k = n \implies j = n+1$. Hence

$$\phi_{n+1}(t) = \frac{-1^0 t^(2\dot{0})}{0!} + \sum_{j=1}^{n+1} \frac{(-1)^j}{(j)!} t^{2j} = \sum_{j=0}^{n+1} \frac{(-1)^j t^{2j}}{j!} \tag{11.52}$$

which is identical to (11.43). This means that our hypothesis, given by equation (11.42), is correct:

$$\phi_n(t) = \sum_{k=0}^{n} \frac{(-1)^k t^{2k}}{k!} \tag{11.53}$$

Our next question is this: does the series

$$\phi(t) = \lim_{n\to\infty} \phi_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!} = \sum_{k=0}^{\infty} \frac{(-t^2)^k}{k!} \tag{11.54}$$

converge? If the answer is yes, then the integral equation, and hence the IVP, has a solution given by $\phi(t)$. Fortunately equation (11.54) resembles a Taylor series that we know from calculus:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{11.55}$$

Comparing the last two equations we conclude that the series does, in fact, converge, and that

$$\phi(t) = e^{-t^2} \tag{11.56}$$

Our final step is to verify that the solution found in this way actually works (because we haven't actually stated any theorems that say the method works!). First we observe that

$$\frac{d}{dt}\phi(t) = \frac{d}{dt}e^{-t^2} = -2te^{-t^2} = -2t\phi(t) \tag{11.57}$$

which agrees with the first line of equation (11.21). To verify the initial condition we can calculate

$$\phi(0) = e^{-(0)^2} = 1 \tag{11.58}$$

as required. □

Lesson 12

Existence of Solutions*

In this section we will prove the fundamental existence theorem. We will defer the proof of the uniqueness until section 13. Since we will prove the theorem using the method of successive approximations, the following statement of the fundamental existence theorem is really just a re-wording of theorem 11.2 with the references to uniqueness and Picard iteration removed. By proving that the Picard iterations converge to the solution, we will, in effect, be proving that a solution exists, which is why the reference to Picard iteration is removed.

Theorem 12.1 (Fundamental Existence Theorem). Suppose that $f(t, y)$ and $\partial f(t, y)/\partial y$ are continuous in some rectangle R defined by

$$t_0 - a \leq t \leq t_0 + a \tag{12.1}$$

$$y_0 - b \leq y \leq y_0 + b \tag{12.2}$$

Then the initial value problem

$$\left.\begin{aligned} \frac{dy}{dt} &= f(t, y) \\ y(t_0) &= y_0 \end{aligned}\right\} \tag{12.3}$$

has a solution in some interval

$$t_0 - a \leq t_0 - h \leq t \leq t_0 + h \leq t_0 + a \tag{12.4}$$

*Most of the material in this section can be omitted without loss of continuity with the remainder of the notes. Students should nevertheless familiarize themselves with the statement of theorem 12.1.

Results from Calculus and Analysis. We will need to use several results from Calculus in this section. These are summarized here for review.

- ***Fundamental Theorem of Calculus.***

  1. $\dfrac{d}{dt}\displaystyle\int_a^t f(s)ds = f(t)$
  2. $\displaystyle\int_a^b \frac{d}{ds}f(s)ds = f(b) - f(a)$

- ***Boundedness Theorem***. Suppose $|f| < M$ and $a < b$. Then

  1. $\displaystyle\int_a^b f(t)dt \leq M(b-a)$
  2. $\displaystyle\int_a^b f(t)dt \leq \left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$

- ***Mean Value Theorem***. If $f(t)$ is differentiable on $[a,b]$ then there is some number $c \in [a,b]$ such that $f(b) - f(a) = f'(c)(b-a)$.

- ***Pointwise Convergence***. Let $A, B \subset \mathbb{R}$, and suppose that $f_k(t) : A \to B$ for $k = 0, 1, 2, ...$ Then the sequence of functions f_k,$k = 0, 1, 2, ...$ is said to converge pointwise to $f(t)$ if for every $t \in A$, $\lim_{k\to\infty} f_k(t) = f(t)$, and we write this as $f_k(t) \to f(t)$.

- ***Uniform Convergence***. The sequence of functions f_k,$k = 0, 1, 2, ...$ is said to converge uniformly to $f(t)$ if for every $\varepsilon > 0$ there exists an integer N such that for every $k > N$, $|f_k(t) - f(t)| < \varepsilon$ for all $t \in A$. Furthermore, If $f_k(t)$ is continuous and$f_k(t) \to f(t)$ uniformly, then

  1. $f(t)$ is continuous.
  2. $\lim_{k\to\infty} \int_a^b f_k(t)dt = \int_a^b \lim_{k\to\infty} f_k(t)dt = \int_a^b f(t)dt$

- ***Pointwise Convergence of a Series***. The series $\sum_{k=0}^{\infty} f_k(t)$ is said to converge pointwise to $s(t)$ if the sequence of partial sums $s_n(t) = \sum_{k=0}^{n} f_k(t)$ converges pointwise to $s(t)$, and we write this as $\sum_{k=0}^{\infty} f_k(t) = s(t)$.

- ***Uniform convergence of a Series.*** The series $\sum_{k=0}^{\infty} f_k(t)$ is said to converge uniformly to $s(t)$ if the sequence of partial sums $s_n(t) = \sum_{k=0}^{n} f_k(t)$ converges uniformly to $s(t)$.

- ***Interchangeability of Limit and Summation.*** If $\sum_{k=0}^{\infty} f_k(t)$ converges uniformly to $s(t)$: $\lim_{t\to a} \sum_{k=0}^{\infty} f_k(t) = \sum_{k=0}^{\infty} \lim_{t\to a} f_k(t) = \sum_{k=0}^{\infty} f(a) = s(a)$. In other words, the limit of the sum is the sum of the limits. 5

Approach The method we will use to prove 12.1 is really a formalization of the method we used in example 11.1:

1. Since f is continuous in the box R it is defined at every point in R, hence it must be bounded by some number M. By theorem (12.3), f is Lipshitz (Definition (12.2)). In Lemma (12.4) use this observation to show that the Picard iterates exist in R and satisfy

$$|\phi_k(t) - y_0| \le M|t - t_0| \tag{12.5}$$

2. Defining $s_n(t) = [\phi_n(t) - \phi_{n-1}(t)]$, we will show in Lemma (12.5) that the sequence $\phi_0, \phi_1, \phi_2, \ldots$ converges if and only if the series

$$S(t) = \sum_{n=1}^{\infty} s_n(t) \tag{12.6}$$

also converges.

3. Use the Lipshitz condition to prove in Lemma (12.6) that

$$|s_n(t)| = |\phi_n - \phi_{n-1}| \le K^{n-1} M \frac{(t - t_0)^n}{n!} \tag{12.7}$$

4. In Lemma (12.7), use equation (12.7) to show that S, defined by (12.6), converges by comparing it with the Taylor series for an exponential, and hence, in Lemma (12.8), that the sequence $\phi_0, \phi_1, \phi_2, \ldots$ also converges to some function $\phi = \lim_{n\to\infty} \phi_n$.

5. In Lemma (12.9), show that $\phi = \lim_{n\to\infty} \phi_n$ is defined and continuous on the rectangle R.

6. Show that $\phi(t)$ satisfies the initial value problem (12.3),

Assumptions. We will make the following assumptions for the rest of this section.

1. R is a rectangle of width $2a$ and height $2b$, centered at (t_0, y_0). Equations (12.1) and (12.2) follow as a consequence.

2. $f(t, y)$ is bounded by some number M on a rectangle R, i.e,

$$|f(t, y)| < M \tag{12.8}$$

   for all t, y in R.

3. $f(t, y)$ is differentiable on R.

4. $\partial f/\partial y$ is also bounded by M on R, i.e.,

$$\left|\frac{\partial f(t, y)}{\partial y}\right| < M \tag{12.9}$$

5. Define the Picard iterates as $\phi_0(t) = y_0$ and

$$\phi_k(t) = y_0 + \int_{t_0}^{t} f(s, \phi_{k-1}(s))ds \tag{12.10}$$

   for all $k = 1, 2, \ldots$.

Definition 12.2. Lipshitz Condition. A function $f(t, y)$ is said to satisfy a *Lipshitz Condition on y in the rectangle R* or be *Lipshitz in y on R* if there exists some number $K > 0$, that we will call the **Lipshitz Constant**, such that

$$|f(t, y_1) - f(t, y_2)| \leq K|y_1 - y_2| \tag{12.11}$$

for all t, y_1, y_2 in some rectangle R.

Example 12.1. Show that $f(t, y) = ty^2$ is Lipshitz on the square $-1 < t < 1, -1 < y < 1$.

We need to find a K such that

$$|f(t, p) - f(t, q)| \leq K|p - q| \tag{12.12}$$

for $f(t, y) = ty^2$, i.e., we need to show

$$|tp^2 - tq^2| \leq K|p - q| \tag{12.13}$$

But

$$|tp^2 - tq^2| = |t(p - q)(p + q)| \tag{12.14}$$

so we need to find a K such that

$$|t(p - q)(p + q)| \leq K|p - q| \tag{12.15}$$

$$|t||p + q| \leq K \tag{12.16}$$

But on the square $-1 \leq t \leq 1, -1 \leq y \leq 1$,

$$|t||p+q| \leq 1 \times 2 = 2 \tag{12.17}$$

So we need to find a $K \geq 2$. We can pick any such K, e.g., $K = 2$. Then every stop follows as a consequence reading from the bottom (12.16) to the top (12.13). Hence f is Lipshitz. □

Theorem 12.3. Boundedness $\implies$ Lipshitz. If $f(t, y)$ is continuously differentiable and there exists some positive number K such that

$$\left|\frac{\partial f}{\partial y}\right| < K \tag{12.18}$$

for all $y, y \in R$ (for some rectangle R), then f is Lipshitz in y on R with Lipshitz constant K.

Proof. By the mean value theorem, for any p, q, there is some number c between p and q such that

$$\frac{\partial}{\partial y} f(t, c) = \frac{f(t, p) - f(t, q)}{p - q} \tag{12.19}$$

By the assumption (12.18),

$$\left|\frac{f(t, p) - f(t, q)}{p - q}\right| < K \tag{12.20}$$

hence

$$|f(t, p) - f(t, q)| < K|p - q| \tag{12.21}$$

for all p, q, which is the definition of Lipshitz. Hence f is Lipshitz. □

Example 12.2. Show that $f(t, y) = ty^2$ is Lipshitz in $-1 < t < 1, -1 < y < 1$.

Since

$$\left|\frac{\partial f}{\partial y}\right| = |2ty| \leq 2 \times 1 \leq 1 = 2 \tag{12.22}$$

on the square, the function is Lipshitz with $K = 2$. □

Lemma 12.4. If f is Lipshitz in y with Lipshitz constant K, then each of the $\phi_i(t)$ are defined on R and satisfy

$$|\phi_k(t) - y_0| \leq M|t - t_0| \tag{12.23}$$

Proof. For $k = 0$, equation says

$$|\phi_0(t) - y_0| \leq M|t - t_0| \tag{12.24}$$

Since $\phi_0(t) = y_0$, the left hand side is zero, so the right hands side, being an absolute value, is ≥ 0.

For $k > 0$, prove the Lemma inductively. Assume that (12.23) is true and use this prove that

$$|\phi_{k+1}(t) - y_0| \leq M|t - t_0| \tag{12.25}$$

From the definition of ϕ_k (see equation (12.10)),

$$\begin{aligned}
\phi_{k+1}(t) &= y_0 + \int_{t_0}^{t} f(s, \phi_k(s))ds && (12.26)\\
|\phi_{k+1}(t) - y_0| &= \left| \int_{t_0}^{t} f(s, \phi_k(s))ds \right| && (12.27)\\
&\leq \int_{t_0}^{t} |f(s, \phi_k(s))|ds && (12.28)\\
&\leq \int_{t_0}^{t} M ds && (12.29)\\
&= M|t - t_0| && (12.30)
\end{aligned}$$

which proves equation (12.25). □

Lemma 12.5. The sequence $\phi_0, \phi_1, \phi_2, \ldots$ converges if and only if the series

$$S(t) = \sum_{k=1}^{\infty} s_n(t) \tag{12.31}$$

also converges, where

$$s_n(t) = [\phi_n(t) - \phi_{n-1}(t)] \tag{12.32}$$

Proof.

$$\begin{aligned}
\phi_n(t) &= \phi_0 + (\phi_1 - \phi_0) + (\phi_2 - \phi_1) + \cdots + (\phi_n - \phi_{n-1}) && (12.33)\\
&= \phi_0 + \sum_{k=1}^{n} (\phi_k - \phi_{k-1}) && (12.34)\\
&= \phi_0 + \sum_{k=1}^{n} s_n(t) && (12.35)
\end{aligned}$$

Thus

$$\lim_{n\to\infty} \phi_n = \phi_0 + \lim_{n\to\infty} \sum_{k=1}^{n} s_n(t) = \phi_0 + \sum_{k=1}^{\infty} s_n(t) = \phi_0 + S(t) \tag{12.36}$$

The left hand side of the equation exists if and only if the right hand side of the equation exists. Hence $\lim_{n\to\infty} \phi_n$ exists if and only if $S(t)$ exists, i.e, $S(t)$ converges. □

Lemma 12.6. With s_n, K, and M as previously defined,

$$|s_n(t)| = |\phi_n - \phi_{n-1}| \le K^{n-1} M \frac{|t-t_0|^n}{n!} \tag{12.37}$$

Proof. For $n = 1$, equation (12.37) says that

$$|\phi_1 - \phi_0| \le K^0 M \frac{|t-t_0|^1}{1!} = M|t-t_0| \tag{12.38}$$

We have already proven that this is true in Lemma (12.4).

For $n > 1$, prove inductively. Assume that (12.37) is true and then prove that

$$|\phi_{n+1} - \phi_n| \le K^n M \frac{|t-t_0|^{n+1}}{(n+1)!} \tag{12.39}$$

Using the definition of ϕ_n and ϕ_{n+1},

$$\begin{align}
|\phi_{n+1} - \phi_n| &= \left| y_0 + \int_{t_0}^{t} f(s, \phi_{n+1}(s))ds - y_0 - \int_{t_0}^{t} f(s, \phi_n(s))ds \right| \tag{12.40}\\
&= \left| \int_{t_0}^{t} [f(s, \phi_{n+1}(s))ds - f(s, \phi_n(s))]ds \right| \tag{12.41}\\
&\le \int_{t_0}^{t} |f(s, \phi_{n+1}(s))ds - f(s, \phi_n(s))|ds \tag{12.42}\\
&\le \int_{t_0}^{t} K|\phi_{n+1}(s) - \phi_n(s)|ds \tag{12.43}
\end{align}$$

where the last step follows because f is Lipshitz in y. Substituting equation (12.37) gives

$$\begin{align}
|\phi_{n+1} - \phi_n| &\le K \int_{t_0}^{t} K^{n-1} M \frac{|s-t_0|^n}{n!} ds \tag{12.44}\\
&= \frac{K^n M}{n!} \int_{t_0}^{t} |s-t_0|^n ds \tag{12.45}\\
&= \frac{K^n M}{n!} \frac{|t-t_0|^{n+1}}{n+1} \tag{12.46}
\end{align}$$

which is what we wanted to prove (equation (12.39)). □

Lemma 12.7. The series $S(t)$ converges.

Proof. By Lemma (12.6)

$$\begin{aligned}
\sum_{n=1}^{\infty} |\phi_n(t) - \phi(t)| &\leq \sum_{n=1}^{\infty} K^{n-1} M \frac{|t-t_0|^n}{n!} && (12.47)\\
&= \frac{M}{K} \sum_{n=1}^{\infty} \frac{|K(t-t_0)|^n}{n!} && (12.48)\\
&= \frac{M}{K} \left(\sum_{n=0}^{\infty} \frac{|K(t-t_0)|^n}{n!} - 1 \right) && (12.49)\\
&= \frac{M}{K} \left(e^{K|t-t_0|} - 1 \right) && (12.50)\\
&\leq \frac{M}{K} e^{K|t-t_0|} && (12.51)
\end{aligned}$$

Since each term in the series for S is absolutely bounded by the corresponding term in the power series for the exponential, the series S converges absolutely, hence it converges. □

Lemma 12.8. The sequence $\phi_0, \phi_1, \phi_2, \ldots$ converges to some limit $\phi(t)$.

Proof. Since the series for $S(t)$ converges, then by Lemma (12.5), the sequence $\phi_0, \phi_1, \ldots$ converges to some function $\phi(t)$. □

Lemma 12.9. $\phi(t)$ is defined and continous on R.

Proof. For any s, t,

$$\begin{aligned}
|\phi_n(s) - \phi_n(t)| &= \left| \int_{t_0}^{s} f(x, \phi_n(x))dx - \int_{t_0}^{t} f(x, \phi_n(x))dx \right| && (12.52)\\
&= \left| \int_{t}^{s} f(x, \phi_n(s))dx \right| && (12.53)\\
&\leq \left| \int_{t}^{s} M dx \right| && (12.54)\\
&= M|s-t| && (12.55)
\end{aligned}$$

Hence taking the limit,

$$\lim_{n\to\infty} |\phi_n(s) - \phi_n(t)| \leq M|s-t| \tag{12.56}$$

because the right hand side does not depend on n. But since $\phi_n \to \phi$, the left hand side becomes $|\phi(s) - \phi(t)|$, i.e,

$$|\phi(s) - \phi(t)| \le M|s - t| \tag{12.57}$$

To show that $\phi(t)$ is continuous we need to show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|s - t| < \delta$, then $|\phi(s) - \phi(t)| < \epsilon$.

Let $\epsilon > 0$ be given and define $\delta = \epsilon/M$. Then

$$|s - t| < \delta \implies |\phi(s) - \phi(t)| \le M|s - t| \le M\delta = \epsilon \tag{12.58}$$

as required. Hence $\phi(t)$ is continuous. $\square$

Proof of the Fundamental Existence Theorem (theorem (12.1)). We have already shown that the sequence $\phi_n \to \phi$ converges to a continuous function on R. To prove the existence theorem we need only to show that ϕ satisfies the initial value problem (12.3), or equivalently, the integral equation

$$\phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))ds \tag{12.59}$$

Let us define the function

$$F(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))ds \tag{12.60}$$

Since $F(t_0) = y_0$, F satisfies the initial condition, and since

$$F'(t) = f(t, \phi(t)) = \phi'(t) \tag{12.61}$$

F also satisfies the differential equation. If we can show that $F(t) = \phi(t)$ then we have shown that ϕ solves the IVP.

We consider the difference

$$|F(t) - \phi_{n+1}(t)| = \left|y_0 + \int_{t_0}^{t} f(s, \phi(s))ds - y_0 - \int_{t_0}^{t} f(s, \phi_n(s))ds\right| \tag{12.62}$$

$$= \left|\int_{t_0}^{t} \left(f(s, \phi(s)) - f(s, \phi_n(s))\right) ds\right| \tag{12.63}$$

$$\le \int_{t_0}^{t} |f(s, \phi(s)) - f(s, \phi_n(s))| \, ds \tag{12.64}$$

$$\le K \int_{t_0}^{t} |\phi(s) - \phi_n(s)| \, ds \tag{12.65}$$

where the last step follows from the Lipshitz condition. Taking limits

$$\lim_{n\to\infty} |F(t) - \phi_{n+1}(t)| \leq \lim_{n\to\infty} K \int_{t_0}^{t} |\phi(s) - \phi_n(s)|\, ds \tag{12.66}$$

But since $\phi_n \to \phi$,

$$|F(t) - \phi(t)| \leq K \int_{t_0}^{t} \big|\phi(s) - \phi_(s)\big|\, ds = 0 \tag{12.67}$$

Since the left hand side is an absolute value it must also be greater then or equal to zero:

$$0 \leq |F(t) - \phi(t)| \leq 0 \tag{12.68}$$

Hence

$$|F(t) - \phi(t)| = 0 \implies F(t) = \phi(t) \tag{12.69}$$

Thus ϕ satisfies the initial value problem. $\square$

Lesson 13

Uniqueness of Solutions*

Theorem 13.1. Uniqueness of Solutions. Suppose that $y = \phi(t)$ is a solution to the initial value problem

$$\left.\begin{aligned} y'(t) &= f(t, y) \\ y'(0) &= t_0 \end{aligned}\right\} \tag{13.1}$$

where $f(t, y)$ and $\partial f(t, y)/\partial y$ are continuous on a box R defined by

$$t_0 - a \leq t \leq t_0 + a \tag{13.2}$$

$$y_0 - b \leq y \leq y_0 + b \tag{13.3}$$

The the solution $y = \phi(t)$ is unique, i.e., if there is any other solution $y = \psi(t)$ then $\phi(t) = \psi(t)$ for all $t \in R$.

The following example illustrates how a solution might not be unique.

Example 13.1. There is no unique solution to the initial value problem

$$\left.\begin{aligned} y'(t) &= \sqrt{y} \\ y(1) &= 1 \end{aligned}\right\} \tag{13.4}$$

Of course we can *find* a solution - the variables are easily separated,

$$\int y^{-1/2} dy = \int dt \tag{13.5}$$

$$2y^{1/2} = t + C \tag{13.6}$$

$$y = \frac{1}{4}(t + C)^2 \tag{13.7}$$

From the initial condition,

$$1 = \frac{1}{4}(1+C)^2 \tag{13.8}$$

When we solve for C we get two possible values:

$$(1+C)^2 = 4 \tag{13.9}$$

$$1 + C = \pm\sqrt{4} = \pm 2 \tag{13.10}$$

$$C = \pm 2 - 1 = 1 \text{ or } -3 \tag{13.11}$$

Using these in equation (13.7) gives two possible solutions:

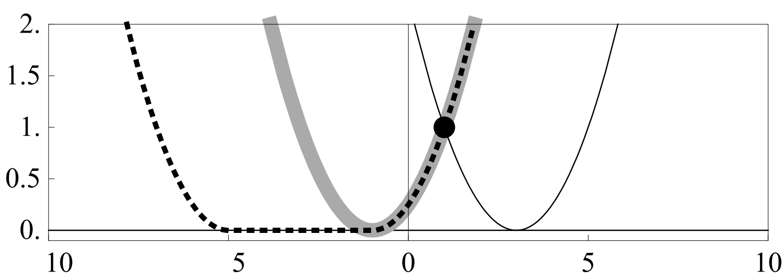
$$y_1 = \frac{1}{4}(t+1)^2 \tag{13.12}$$

$$y_2 = \frac{1}{4}(t-3)^2 \tag{13.13}$$

It is easily verified that both of these satisfy the initial value problem. In fact, there are other solutions that also satisfy the initial value problem, that we cannot obtain by the straightforward method of integration given above. For example, if $a < -1$ then

$$y_a = \begin{cases} \dfrac{1}{4}(t-a)^2, & t \le a \\ 0, & a \le t \le -1 \\ \dfrac{1}{4}(t+1)^2, & t \ge -1 \end{cases} \tag{13.14}$$

is also a solution (you should verify this by (a) showing that it satisfies the initial condition; (b) differentiating each piece and showing that it satisfies the differential equation independently of the other two pieces; and then showing that (c) the function is continuous at $t = a$ and $t = -1$.



The different non-unique solutions are illustrated in the figure above; they all pass through the point (1,1), and hence satisfy the initial condition. □

The proof of theorem (13.1) is similar to the following example.

Example 13.2. Show that the initial value problem

$$\left.\begin{aligned} y'(t,y) &= ty \\ y(0) &= 1 \end{aligned}\right\} \tag{13.15}$$

has a unique solution on the interval $[-1, 1]$.

The solution itself is easy to find; the equation is separable.

$$\int \frac{dy}{y} = \int t dt \implies y = Ce^{t^2/2} \tag{13.16}$$

The initial condition tells us that $C = 1$, hence $y = e^{t^2/2}$.

The equivalent integral equation to (13.15) is

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds \tag{13.17}$$

Since $f(t, y) = ty$, $t_0 = 0$, and $y_0 = 1$,

$$y(t) = 1 + \int_0^t sy(s)ds \tag{13.18}$$

Suppose that $z(t)$ is another solution; then $z(t)$ must also satisfy

$$z(t) = 1 + \int_0^t sz(s)ds \tag{13.19}$$

To prove that the solution is unique, we need to show that $y(t) = z(t)$ for all t in the interval $[-1, 1]$. To do this we will consider their difference $\delta(t) = |z(t) - y(t)|$ and show that is must be zero.

$$\begin{aligned}
\delta(t) &= |z(t) - y(t)| && (13.20)\\
&= \left|1 + \int_0^t sz(s)ds - 1 - \int_0^t sy(s)ds\right| && (13.21)\\
&= \left|\int_0^t [sz(s)ds - sy(s)]ds\right| && (13.22)\\
&\leq \int_0^t |s||z(s) - y(s)|ds && (13.23)\\
&\leq \int_0^t |z(s) - y(s)|ds && (13.24)\\
&= \int_0^t \delta(s)ds && (13.25)
\end{aligned}$$

where the next-to-last step follows because inside the integral $|s| < 1$. Next, we define a function $F(t)$ such that

$$F(t) = \int_0^t \delta(s)ds \tag{13.26}$$

Since F is an integral of an absolute value,

$$F(t) \geq 0 \tag{13.27}$$

Then

$$F'(t) = \frac{d}{dt}\int_0^t \delta(s)ds = \delta(t) \tag{13.28}$$

Since by equation (13.25) $\delta(t) \leq F(t)$, we arrive at

$$F'(t) = \delta(t) \leq F(t) \tag{13.29}$$

Therefore

$$F'(t) - F(t) \leq 0 \tag{13.30}$$

From the product rule,

$$\begin{aligned}
\frac{d}{dt}\left[e^{-t}F(t)\right] &= e^{-t}F'(t) - e^{-t}F(t) && (13.31)\\
&= e^{-t}[F'(t) - F(t)] && (13.32)\\
&\leq 0 && (13.33)
\end{aligned}$$

Integrating both sides of the equation from 0 to t,

$$\begin{aligned}
\int_0^t \frac{d}{dt}\left[e^{-s}F(s)ds\right] \leq 0 && (13.34)\\
e^{-t}F(t) - e^0F(0) \leq 0 \text{ (Fund. Thm. of Calc.)} && (13.35)\\
e^{-t}F(t) \leq 0 \ (F(0){=}0) && (13.36)\\
F(t) \leq 0 \text{ (divide by the exponential)} && (13.37)
\end{aligned}$$

Now compare equations (13.27) and (13.37); the only consistent conclusion is that

$$F(t) = 0 \tag{13.38}$$

for all t. Thus

$$\int_0^t \delta(s)ds = 0 \tag{13.39}$$

But $\delta(t)$ is an absolute value, so it can never take on a negative value. The integral is the area under the curve from 0 to t. The only way this area can be zero is if the

$$\delta(t) = 0 \tag{13.40}$$

for all t. Hence

$$z(t) = y(t) \tag{13.41}$$

for all t. Thus the solution is unique. □

Theorem 13.2. Gronwall Inequality. Let f, g be continuous, real functions on some interval $[a, b]$ that satisfy

$$f(t) \leq K + \int_a^t f(s)g(s)ds \tag{13.42}$$

for some constant $K \geq 0$. Then

$$f(t) \leq K\exp\left(\int_a^t g(s)ds\right) \tag{13.43}$$

Proof. Define the following functions:

$$F(t) = K + \int_a^t f(s)g(s) \tag{13.44}$$

$$G(t) = \int_a^t g(s)ds \tag{13.45}$$

Then

$$F(a) = K \tag{13.46}$$

$$G(a) = 0 \tag{13.47}$$

$$F'(t) = f(t)g(t) \tag{13.48}$$

$$G'(t) = g(t) \tag{13.49}$$

By equation (13.42) we are given that

$$f(t) \leq F(t) \tag{13.50}$$

hence from equation (13.48)

$$F'(t) = f(t)g(t) \leq F(t)g(t) = F(t)G'(t) \tag{13.51}$$

where the last step follows from (13.49). Hence

$$F'(t) - F(t)G'(t) \leq 0 \tag{13.52}$$

By the product rule,

$$\frac{d}{dt}\left[F(t)e^{-G(t)}\right] = F'(t)e^{-G(t)} - F(t)G'(t)e^{-G(t)} \tag{13.53}$$
$$= [F'(t) - F(t)G'(t)]e^{-G(t)} \tag{13.54}$$
$$\leq 0 \tag{13.55}$$

Integrating the left hand side of (13.53)

$$\int_a^t \frac{d}{ds}\left[F(s)e^{-G(s)}\right] ds = F(t)e^{-G(t)} - F(a)e^{-G(a)} \tag{13.56}$$
$$= F(t)e^{-G(t)} - K \tag{13.57}$$

Since the integral of a negative function must be negative,

$$F(t)e^{-G(t)} - K \leq 0 \tag{13.58}$$
$$F(t)e^{-G(t)} \leq K \tag{13.59}$$
$$F(t) \leq Ke^{G(t)} \tag{13.60}$$

which is equation (13.43). □

Proof of Theorem (13.1) Suppose that y and z are two different solutions of the initial value problem. Then

$$y(t) = y_0 + \int_{t_0}^t f(s, y(s))ds \tag{13.61}$$
$$z(t) = y_0 + \int_{t_0}^t f(s, z(s))ds \tag{13.62}$$

Therefore

$$|y(t) - z(t)| = \left|\int_{t_0}^t f(s, y(s))ds \; - \int_{t_0}^t f(s, z(s))ds\right| \tag{13.63}$$
$$= \left|\int_{t_0}^t [f(s, y(s)) - f(s, z(s))]ds\right| \tag{13.64}$$
$$\leq \int_{t_0}^t |f(s, y(s)) - f(s, z(s))|ds \tag{13.65}$$

Since $|\partial f/\partial y|$ is continuous on a closed interval it is bounded by some number M, and hence f is Lipshitz with Lipshitz constant M. Thus

$$|f(s, y(s)) - f(s, z(s)| \leq M|y(s) - z(s)| \tag{13.66}$$

Substituting (13.66) into (13.65)

$$|y(t) - z(t)| \leq M \int_{t_0}^{t} |y(s) - z(s)| ds \tag{13.67}$$

Let

$$f(t) = |y(t) - z(t)| \tag{13.68}$$

Then

$$f(t) \leq M \int_{t_0}^{t} f(s) ds \tag{13.69}$$

Then f satisfies the condition for the Gronwall inequality with $K = 0$ and $g(t) = M$, which means

$$f(t) \leq K \exp \int_{a}^{t} g(s) ds = 0 \tag{13.70}$$

Since $f(t)$ is an absolute value it can never be negative so it must be zero.

$$0 = f(t) = |y(t) - z(t)| \tag{13.71}$$

for all t. Hence

$$y(t) = z(t) \tag{13.72}$$

for all t. Thus any two solutions are identical, i.e, the solution is unique. $\square$

ϕ(t) = y0 + ∫ f(s, ϕ(s)) ds
N(t,y) = ∂ϕ/∂y
differentiable
and |∂f/∂y| ≤ K on R then
f is Lipschitz on R

Lesson 14

Review of Linear Algebra

In this section we will recall some concepts from linear algebra class

Definition 14.1. A **Euclidean 3-vector v** is object with a **magnitude** and **direction** which we will denote by the ordered triple

$$\mathbf{v} = (x, y, z) \tag{14.1}$$

The **magnitude** or **absolute value** or **length** of the v is denoted by the postitive square root

$$v = |\mathbf{v}| = \sqrt{x^2 + y^2 + z^2} \tag{14.2}$$

This definition is motivated by the fact that v is the length of the line segment from the origin to the point $P = (x, y, z)$ in Euclidean 3-space.

A vector is sometimes represented geometrically by an arrow from the origin to the point $P = (x, y, z)$, and we will sometimes use the notation (x, y, z) to refer either to the point P or the vector $\mathbf{v}$ from the origin to the point P. Usually it will be clear from the context which we mean. This works because of the following theorem.

Definition 14.2. The set of all Euclidean 3-vectors is isomorphic to the Euclidean 3-space (which we typically refer to as $\mathbb{R}^3$).

If you are unfamiliar with the term *isomorphic*, don't worry about it; just take it to mean "in one-to-one correspondence with," and that will be sufficient for our purposes.

Definition 14.3. Let $\mathbf{v} = (x, y, z)$ and $\mathbf{w} = (x', y', z')$ be Euclidean 3-vectors. Then the **angle between v and w** is defined as the angle between the line segments joining the origin and the points $P = (x, y, z)$ and $P' = (x', y', z')$.

We can define **vector addition** or **vector subtraction** by

$$\mathbf{v} + \mathbf{w} = (x, y, z) + (x', y', z') = (x + x', y + y', z + z') \tag{14.3}$$

where $\mathbf{v} = (x, y, z)$ and $\mathbf{w} = (x', y', z')$, and **scalar multiplcation** (multiplication by a real number) by

$$k\mathbf{v} = (kx, ky, kz) \tag{14.4}$$

Theorem 14.4. The set of all Euclidean vectors is closed under vector addition and scalar multiplication.

Definition 14.5. Let $\mathbf{v} = (x, y, z), \mathbf{w} = (x', y', z')$ be Euclidean 3-vectors. Their **dot product** is defined as

$$\mathbf{v} \cdot \mathbf{w} = xx' + yy' + zz' \tag{14.5}$$

Theorem 14.6. Let θ be the angle between the line segments from the origin to the points (x, y, z) and (x', y', z') in Euclidean 3-space. Then

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos \theta \tag{14.6}$$

Definition 14.7. The **standard basis vectors** for Euclidean 3-space are the vectors

$$\mathbf{i} = (1, 0, 0) \tag{14.7}$$

$$\mathbf{j} = (0, 1, 0) \tag{14.8}$$

$$\mathbf{k} = (0, 0, 1) \tag{14.9}$$

Theorem 14.8. Let $\mathbf{v} = (x, y, z)$ be any Euclidean 3-vector. Then

$$\mathbf{v} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z \tag{14.10}$$

Definition 14.9. The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are said to be **linearly dependent** if there exist numbers $a_1, a_2, \ldots, a_n$, not all zero, such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = 0 \tag{14.11}$$

If no such numbers exist the vectors are said to be **linearly independent**.

Definition 14.10. An $m \times n$ (or m by n) **matrix** A is a rectangular array of number with m rows and n columns. We will denote the number in the i^{th} row and j^{th} column as a_{ij}

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \tag{14.12}$$

We will sometimes denote the matrix A by $[a_{ij}]$.

The **transpose** of the matrix A is the matrix obtained by interchanging the row and column indices,

$$(A^{\mathrm{T}})_{ij} = a_{ji} \tag{14.13}$$

or

$$[a_{ij}]^{\mathrm{T}} = [a_{ji}] \tag{14.14}$$

The transpose of an $m \times n$ matrix is an $n \times m$ matrix. We will sometimes represent the vector $\mathbf{v} = (x, y, z)$ by its 3×1 **column-vector** representation

$$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{14.15}$$

or its 1×3 **row-vector** representation

$$\mathbf{v}^{\mathrm{T}} = \begin{pmatrix} x & y & z \end{pmatrix} \tag{14.16}$$

Definition 14.11. **Matrix Addition** is defined between two matrices of the same size, by adding corresponding elemnts.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & & \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots \\ b_{21} & b_{22} & \cdots \\ \vdots & & \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & b_{22} + b_{12} & \cdots \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots \\ \vdots & & \end{pmatrix} \tag{14.17}$$

Matrices that have different sizes cannot be added.

Definition 14.12. A **square matrix** is any matrix with the same number of rows as columns. The **order** of the square matrix is the number of rows (or columns).

Definition 14.13. Let A be a square matrix. A **submatrix** of A is the matrix A with one (or more) rows and/or one (or more) columns deleted.

Definition 14.14. The **determinant of a square matrix** is defined as follows. Let A be a square matrix and let n be the order of A. Then

1. If $n = 1$ then $A = [a]$ and $\det A = a$.
2. If $n \geq 2$ then

$$\det A = \sum_{i=1}^{n} a_{ki}(-1)^{i+k} \det(A'_{ik}) \tag{14.18}$$

for any $k = 1, .., n$, where by A'_{ik} we mean the submatrix of A with the i^{th} row and k^{th} column deleted. (The choice of which k does not matter because the result will be the same.)

We denote the determinant by the notation

$$detA = \begin{vmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & & \end{vmatrix} \tag{14.19}$$

In particular,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{14.20}$$

and

$$\begin{vmatrix} A & B & C \\ D & E & F \\ G & H & I \end{vmatrix} = A\begin{vmatrix} E & F \\ H & I \end{vmatrix} - B\begin{vmatrix} D & F \\ G & I \end{vmatrix} + C\begin{vmatrix} D & E \\ G & H \end{vmatrix} \tag{14.21}$$

Definition 14.15. Let $v = (x, y, z)$ and $w = (x', y', z')$ be Euclidean 3-vectors. Their **cross product** is

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ x' & y' & z \end{vmatrix} = (yz' - y'z)\mathbf{i} - (xz' - x'z)\mathbf{j} + (xy' - x'y)\mathbf{k} \tag{14.22}$$

Theorem 14.16. Let $v = (x, y, z)$ and $w = (x', y', z')$ be Euclidean 3-vectors, and let θ be the angle between them. Then

$$|\mathbf{v} \times \mathbf{v}| = |\mathbf{v}||\mathbf{w}| \sin\theta \tag{14.23}$$

Definition 14.17. A square matrix A is said to be **singular** if $\det A = 0$, and **non-singular** if $\det A \neq 0$.

Theorem 14.18. The n columns (or rows) of an $n \times n$ square matrix A are linearly independent if and only if $\det A \neq 0$.

Definition 14.19. Matrix Multiplication. Let $A = [a_{ij}]$ be an $m \times r$ matrix and let $B = [b_{ij}]$ be an $r \times n$ matrix. Then the matrix product is defined by

$$[AB]_{ij} = \sum_{k=1}^{r} a_{ik} b_{kr} = \text{row}_i(A) \cdot \text{column}_j B \tag{14.24}$$

i.e., the ij^{th} element of the product is the dot product between the i^{th} row of A and the j^{th} column of B.

Example 14.1.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 8 & 9 \\ 10 & 11 \\ 12 & 13 \end{pmatrix} = \begin{pmatrix} (1,2,3)\cdot(8,10,12) & (1,2,3)\cdot(9,11,13) \\ (4,5,6)\cdot(8,10,12) & (4,5,6)\cdot(9,11,13) \end{pmatrix} \tag{14.25}$$

$$= \begin{pmatrix} 64 & 70 \\ 156 & 169 \end{pmatrix} \quad \square \tag{14.26}$$

Note that the product of an $[n \times r]$ matrix and an $[r \times m]$ matrix is always an $[n \times m]$ matrix. The product of an $[n \times r]$ matrix and and $[s \times n]$ is undefined unless $r = s$.

Theorem 14.20. If A and B are both $n \times n$ square matrices then

$$\det AB = (\det A)(\det B) \tag{14.27}$$

Definition 14.21. Identity Matrix. The $n \times n$ matrix I is defined as the matrix with 1's in the **main diagonal** $a_{11}, a_{22}, \dots, a_{mm}$ and zeroes everywhere else.

Theorem 14.22. I is the identity under matrix multiplication. Let A be any $n \times n$ matrix and I the $n \times n$ Identity matrix. Then $AI = IA = A$.

Definition 14.23. A square matrix A is said to be **invertible** if there exists a matrix A^{-1}, called the **inverse** of A, such that

$$AA^{-1} = A^{-1}A = I \tag{14.28}$$

Theorem 14.24. A square matrix is invertible if and only if it is nonsingular, i.,e, $\det A \neq 0$.

Definition 14.25. Let $A = [a_{ij}]$ be any square matrix of order n. Then the **cofactor** of a_{ij}, denoted by cof a_{ij}, is the $(-1)^{i+j} \det M_{ij}$ where M_{ij} is the submatrix of A with row i and column j removed.

Example 14.2. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \tag{14.29}$$

Then

$$\text{cof } a_{12} = (-1)^{1+2} \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = (-1)(36-42) = 6 \quad \square \tag{14.30}$$

Definition 14.26. Let A be a square matrix of order n. The **Clasical Adjoint** of A, denoted adj A, is the transopose of the matrix that results when every element of A is replaced by its cofactor.

Example 14.3. Let

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 4 & 5 & 0 \\ 0 & 3 & 1 \end{pmatrix} \tag{14.31}$$

The classical adjoint is

$$\begin{aligned} \text{adj } A = \text{Transpose} & \\ \begin{pmatrix} (1)[(1)(5)-(0)(3)] & (-1)[(4)(1)-(0)(0)] & (1)[(4)(3)-(5)(0)] \\ (-1)[(0)(1)-(3)(3)] & (1)[(1)(1)-(3)(0)] & (-1)[(1)(3)-(0)(0)] \\ (1)[(0)(0)-(3)(5)] & (-1)[(1)(0)-(3)(4)] & (1)[(1)(5)-(0)(4)] \end{pmatrix} & \end{aligned} \tag{14.32}$$

$$= \text{Transpose} \begin{pmatrix} 5 & -4 & 12 \\ 9 & 1 & -3 \\ -15 & 12 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 9 & -15 \\ -4 & 1 & 12 \\ 12 & -3 & 5 \end{pmatrix} \quad \square \tag{14.33}$$

Theorem 14.27. Let A be a non-singular square matrix. Then

$$A^{-1} = \frac{1}{\det A} \text{adj } A \tag{14.34}$$

Example 14.4. Let A be the square matrix defined in equation 14.31. Then

$$\det A = 1(5-0) - 0 + 3(12-0) = 41 \tag{14.35}$$

Hence

$$A^{-1} = \frac{1}{41} \begin{pmatrix} 5 & 9 & -15 \\ -4 & 1 & 12 \\ 12 & -3 & 5 \end{pmatrix} \quad \square \tag{14.36}$$

In practical terms, computation of the determinant is computationally inefficient, and there are faster ways to calculate the inverse, such as via Gaussian Elimination. In fact, determinants and matrix inverses are very rarely used computationally because there is almost always a better way to solve the problem, where by better we mean the total number of computations as measure by number of required multiplications and additions.

Definition 14.28. Let A be a square matrix. Then the **eigenvalues** of A are the numbers λ and **eigenvectors** $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda\mathbf{v} \tag{14.37}$$

Definition 14.29. The **characteristic equation** of a square matrix of order n is the n^{th} order (or possibly lower order) polynomial

$$\det(A - \lambda I) = 0 \tag{14.38}$$

Example 14.5. Let A be the square matrix defined in equation 14.31. Then its characteristic equation is

$$\begin{aligned} 0 &= \begin{vmatrix} 1-\lambda & 0 & 3 \\ 4 & 5-\lambda & 0 \\ 0 & 3 & 1-\lambda \end{vmatrix} && (14.39) \\ &= (1-\lambda)(5-\lambda)(1-\lambda) - 0 + 3(4)(3) && (14.40) \\ &= 41 - 11\lambda + 7\lambda^2 - \lambda^3 \quad \square && (14.41) \end{aligned}$$

Theorem 14.30. The eigenvalues of a square matrix A are the roots of its characteristic polynomial.

Example 14.6. Let A be the square matrix defined in equation 14.31. Then its eigenvalues are the roots of the cubic equation

$$41 - 11\lambda + 7\lambda^2 - \lambda^3 = 0 \tag{14.42}$$

The only real root of this equation is approximately $\lambda \approx 6.28761$. There are two additional complex roots, $\lambda \approx 0.356196 - 2.52861i$ and $\lambda \approx 0.356196 + 2.52861i$.

Example 14.7. Let

$$A = \begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix} \tag{14.43}$$

Its characteristic equation is

$$0 = \begin{vmatrix} 2-\lambda & -2 & 3 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} \tag{14.44}$$

$$= (2-\lambda)[(1-\lambda)(-1-\lambda)-3] + 2[(-1-\lambda)-1] \tag{14.45}$$

$$+ 3[3-(1-\lambda)] \tag{14.46}$$

$$= (2-\lambda)(-1+\lambda^2-3) + 2(-2-\lambda) + 3(2+\lambda) \tag{14.47}$$

$$= (2-\lambda)(\lambda^2-4) - 2(\lambda+2) + 3(\lambda+2) \tag{14.48}$$

$$= (2-\lambda)(\lambda+2)(\lambda-2) + (\lambda+2) \tag{14.49}$$

$$= (\lambda+2)[(2-\lambda)(\lambda-2)+1] \tag{14.50}$$

$$= (\lambda+2)(-\lambda^2+4\lambda-3) \tag{14.51}$$

$$= -(\lambda+2)(\lambda^2-4\lambda+3) \tag{14.52}$$

$$= -(\lambda+2)(\lambda-3)(\lambda-1) \tag{14.53}$$

Therefore the eigenvalues are -2, 3, 1. To find the eigenvector corresponding to -2 we would solve the system of

$$\begin{pmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{14.54}$$

for x, y, z. One way to do this is to multiply out the matrix on the left and solve the system of three equations in three unknowns:

$$2x - 2y + 3z = -2x \tag{14.55}$$

$$x + y + z = -2y \tag{14.56}$$

$$x + 3y - z = -2z \tag{14.57}$$

However, we should observe that the eigenvector is never unique. For example, if v is an eigenvector of A with eigenvalue λ then

$$A(k\mathbf{v}) = kA\mathbf{v} = k\lambda\mathbf{v} \tag{14.58}$$

i.e., $k\mathbf{v}$ is also an eigenvalue of A. So the problem is simplified: we can try to fix one of the elements of the eigenvalue. Say we try to find an eigenvector of A corresponding to $\lambda = -2$ with $y = 1$. Then we solve the system

$$2x - 2 + 3z = -2x \tag{14.59}$$

$$x + 1 + z = -2 \tag{14.60}$$

$$x + 3 - z = -2z \tag{14.61}$$

Simplifying

$$4x - 2 + 3z = 0 \tag{14.62}$$

$$x + 3 + z = 0 \tag{14.63}$$

$$x + 3 + z = 0 \tag{14.64}$$

The second and third equations are now the same because we have fixed one of the values. The remaining two equations give two equations in two unknowns:

$$4x + 3z = 2 \tag{14.65}$$

$$x + z = -3 \tag{14.66}$$

The solution is $x = 11, z = -14$. Therefore an eigenvalue of A corresponding to $\lambda = -2$ is $\mathbf{v} = (11, 1, -14)$, as is any constant multiple of this vector. □

Definition 14.31. The **main diagonal** of a square matrix A is the list $(a_{11}, a_{22}, \ldots, a_{nn})$.

Definition 14.32. A **diagonal matrix** is a square matrix that only has non-zero entries on the main diagonal.

Theorem 14.33. The eigenvalues of a diagonal matrix are the elements of the diagonal.

Definition 14.34. An **upper (lower) triangular matrix** is a square matrix that only has nonzero entries on or above (below) the main diagonal.

Theorem 14.35. The eigenvalues of an upper (lower) triangular matrix line on the main diagonal.

Lesson 15

Linear Operators and Vector Spaces

Definition 15.1. A **Vector Space** over $\mathbb{R}$[1] is a set $\mathcal{V}$ combined with two operations **addition** (denoted by $x+y, y, y \in \mathcal{V}$) and **scalar**[2] **multiplication** (denoted by $c \times y$ or $cy, x \in \mathbb{R}, y \in \mathcal{V}$). with the following properties:

1. **Closure under Addition and Scalar Multiplication**

$$\begin{aligned} y, z \in \mathcal{V} &\implies y+z \in \mathcal{V} \\ t \in \mathbb{R}, y \in \mathcal{V} &\implies ty \in \mathcal{V} \end{aligned} \tag{15.1}$$

2. **Commutativity of Addition**

$$y, z \in \mathcal{V} \implies y+z = z+y \tag{15.2}$$

3. **Associativity of Addition and Scalar Multiplication**

$$\begin{aligned} w, y, z \in \mathcal{V} &\implies (w+y)+z = w+(y+z) \\ a, b \in \mathbb{R}, y \in \mathcal{V} &\implies (ab)y = a(by) \end{aligned} \tag{15.3}$$

4. **Additive Identity**. There exists a $0 \in \mathcal{V}$ such that

$$y \in \mathcal{V} \implies y+0 = 0+y = y \tag{15.4}$$

[1] This definition generalizes with $\mathbb{R}$ replaced by any field.
[2] A scalar is any real number or any real variable.

5. **Additive Inverse**. For each $y \in \mathcal{V}$ there exists a $-y \in \mathcal{V}$ such that

$$y + (-y) = (-y) + y = 0 \tag{15.5}$$

6. **Multiplicative Identity**. For every $y \in \mathcal{V}$,

$$1 \times y = y \times 1 = y \tag{15.6}$$

7. **Distributive Property**. For every $a, b \in \mathbb{R}$ and every $y, z \in \mathcal{V}$,

$$\begin{aligned} a(y + z) &= ay + az \\ (a + b)y &= ay + by \end{aligned} \tag{15.7}$$

Example 15.1. The usual Euclidean 3D space forms a vector space, where each vector is a triple of numbers corresponding to the coordinates of a point

$$v = (x, y, z) \tag{15.8}$$

If $w = (p, q, r)$ then addition of vectors is defined as

$$v + w = (x + p, y + q, r + z) \tag{15.9}$$

and scalar multiplication is given by

$$av = (ax, ay, az) \tag{15.10}$$

You should verify that all seven properties hold. □

We are particularly interested in the following vector space.

Example 15.2. Let $\mathcal{V}$ be the set of all functions $y(t)$ defined on the real numbers. Then $\mathcal{V}$ is a vector space under the usual definitions of addition of functions and multiplication by real numbers. For example, if f and g are functions in $\mathcal{V}$ then

$$h(t) = f(t) + g(t) \tag{15.11}$$

is also in $\mathcal{V}$, and if c is a real number, then

$$p(t) = cf(t) \tag{15.12}$$

is also in $\mathcal{V}$. To see that the distributive property holds, observe that

$$a(f(t) + g(t)) = af(t) + bg(t) \tag{15.13}$$

$$(a + b)f(t) = af(t) + bft(t) \tag{15.14}$$

You should verify that each of the other six properties hold. □

Example 15.3. By a similar argument as in the previous problem, the set of all functions

$$f(t) : \mathbb{R}^n \to \mathbb{R}^m \tag{15.15}$$

is also a vector space, using the usual definitions of function addition and multiplication by a constant.

Definition 15.2. Let $\mathcal{V}$ be a vector space. Then a **norm** on $\mathcal{V}$ is any function $\|y\| : \mathcal{V} \to \mathbb{R}$ (i.e., it maps every vector y in $\mathcal{V}$ to a real number called $\|y\|$) such that

1. $\|y\| \geq 0$ and $\|y\| = 0 \iff y = 0$.
2. $\|cy\| = |c|\|y\|$ for any real number c, vector y.
3. $\|y + z\| \leq \|y\| + \|z\|$ for any vectors y, z. (**Triangle Inequality**)

A **normed vector space** is a vector space with a norm defined on it.

Example 15.4. In the usual Euclidean vector space, the 2-norm, given by

$$\|v\| = \sqrt{x^2 + y^2 + z^2} \tag{15.16}$$

where the positive square root is used. You probably used this norm in Math 280. □

Example 15.5. Another norm that also works in Euclidean space is called the **sup-norm**, defined by

$$\|v\|_\infty = \max(|x|, |y|, |z|) \tag{15.17}$$

Checking each of the three properties:

1. $\|v\|_\infty$ is an absolute value, so it cannot be negative. It can only be zero if each of the three components $x = y = z = 0$, in which case $v = (0, 0, 0)$ is the zero vector.
2. This follows because

$$\begin{aligned}
\|cv\|_\infty &= \|c(x, y, z)\|_\infty && (15.18)\\
&= \|(cx, cy, cz))\|_\infty && (15.19)\\
&= \max(|cx|, |cy|, |cz|) && (15.20)\\
&= |c| \max(|x|, |y|, |z|) && (15.21)\\
&= |c|\|v\|_\infty && (15.22)
\end{aligned}$$

3. The triangle follows from the properties of real numbers,

$$\begin{aligned} \|v+w\|_\infty &= \|(x+p, y+q, z+r)\| && (15.23)\\ &= \max(|x+p|, |y+q|, |z+r|) && (15.24)\\ &\le \max(|x|+|p|, |y|+|q|, |z|+|r|) && (15.25) \end{aligned}$$

because $|x+p| \le |x|+|p|$, etc.Hence

$$\begin{aligned} \|v+w\|_\infty &\le \max(|x|, |y|, |z|) + \max(|p|, |q|, |r|) && (15.26)\\ &= \|v\|_\infty + \|w\|_\infty && (15.27) \end{aligned}$$

Thus $\|v\|_\infty$ is a norm. □

Example 15.6. Let $\mathcal{V}$ be the set of all functions $y(t)$ defined on the interval $0 \le t \le 1$. Then

$$\|y\| = \left(\int_0^1 |y(t)|^2 dt\right)^{1/2} \tag{15.28}$$

is a norm on $\mathcal{V}$.

The first property follows because the integral of an absolute value is a positive number (the area under the curve) unless $y(t) = 0$ for all t, in which case the area is zero.

The second property follows because

$$\|cy\| = \left(\int_0^1 |cy(t)|^2 dt\right)^{1/2} = |c| \left(\int_0^1 |y(t)|^2 dt\right)^{1/2} = |c|\|y\| \tag{15.29}$$

The third property follows because

$$\begin{aligned} \|y+z\|^2 &= \int_0^1 |y(t)+z(t)|^2 dt && (15.30)\\ &= \int_0^1 |y(t)^2 + 2y(t)z(t) + z(t)^2| dt && (15.31)\\ &\le \int_0^1 |y(t)|^2 dt + 2\int_0^1 |y(t)||z(t)| dt + \int_0^1 |z(t)|^2 dt && (15.32) \end{aligned}$$

By the Cauchy Schwarz Inequality[3]

$$\left|\int_0^1 y(t)z(t)dt\right|^2 \le \left(\int_0^1 |y(t)|^2 dt\right)\left(\int_0^1 |z(t)|^2 dt\right) = \|y\|^2\|z\|^2 \tag{15.33}$$

[3]The Cauchy-Schwarz inequality is a property of integrals that says exactly this formula.

Therefore

$$\|y+z\|^2 \leq \|y\|^2 + 2\|y\|\|z\| + \|z\|^2 = (\|y\| + \|z\|)^2 \tag{15.34}$$

Taking square roots of both sides gives the third property of norms. $\square$

Example 15.7. Let $\mathcal{V}$ be the vector space consisting of integrable functions on an interval (a, b), and let $f \in \mathcal{V}$. Then the **sup-norm** defined by

$$\|f\|_\infty = \sup\{|f(t)| : t \in (a, b)\} \tag{15.35}$$

is a norm.

The first property follows because it is an absolute value. The only way the supremum of a non-negative function can be zero is if the function is identically zero.

The second property follows because $\sup |cf(t)| = |c| \sup |f(t)|$

The third property follows from the triangle inequality for real numbers:

$$|f(t) + g(t)| \leq |f(t)| + |f(t)| \tag{15.36}$$

Hence

$$\|f + g\| = \sup |f(t) + g(t)| \leq \sup |f(t)| + \sup |f(t)| = \|f\| + \|g\| \quad \square \tag{15.37}$$

Definition 15.3. A **Linear Operator** is a function $L : \mathcal{V} \mapsto \mathcal{V}$ whose domain and range are both the same vector space, and which has the following properties:

1. **Additivity**. For all vectors $y, z \in \mathcal{V}$,

$$L(y + z) = L(y) + L(z) \tag{15.38}$$

2. **Homogeneity**. For all numbers a and for all vectors $y \in \mathcal{V}$,

$$L(ay) = aL(y) \tag{15.39}$$

These two properties are sometimes written as

$$L(ay + bz) = aL(y) + bL(z) \tag{15.40}$$

It is common practice to omit the parenthesis when discussing linear operators.

Definition 15.4. If y, z are both elements of a vector space $\mathcal{V}$, and A and B are any numbers, we call

$$w = Ay + Bz \tag{15.41}$$

a **Linear Combination** of y and z

Example 15.8. If $v = (1, 0, 3)$ and $w = (5, -3, 12)$ are vectors in Euclidean 3 space, then for any numbers A and B,

$$u = Av + Bw = A(1, 0, 3) + B(5, -3, 12) = (A + 5B, -3B, 3A + 12B) \tag{15.42}$$

is a linear combination of v and w. □

The closure property of vector spaces is sometimes stated as following: **Any linear combination of vectors is an element of the same vector space.** For example, we can create linear combinations of functions and we know that they are also in the same vector space.

Example 15.9. Let $f(t) = 3\sin t, g(t) = t^2 - 4t$ be functions in the vector space $\mathcal{V}$ of real valued functions. Then if A and B are any real numbers,

$$h(t) = Af(t) + Bg(t) \tag{15.43}$$

$$= A\sin t + B(t^2 - 4t) \tag{15.44}$$

is a linear combination of the functions f and g. Since $\mathcal{V}$ is a vector space, h is also in $\mathcal{V}$. □

Example 15.10. Let $\mathcal{V}$ be the vector space consisting of real functions on the real numbers. Then differentiation, defined by

$$D(y) = \frac{dy(t)}{dt} \tag{15.45}$$

is a linear operator. To see that both properties hold let $y(t)$ and $z(t)$ be functions (e.g., $y, z \in \mathcal{V}$) and let c be a constant. Then

$$D(y + z) = \frac{d(y(t) + z(t))}{dt} = \frac{dy(t)}{dt} + \frac{dz(t)}{dt} = D(y) + D(z) \tag{15.46}$$

$$D(cy) = \frac{d(cy(t))}{dt} = c\frac{dy(t)}{dt} = cD(y) \tag{15.47}$$

Hence D is a linear operator. □

Definition 15.5. Two vectors y, z are called **linearly dependent** if there exists nonzero constants A, B such that

$$Ay + Bz = 0 \tag{15.48}$$

If no such A and B exist, then we say that y and z are **Linearly Independent**.

In Euclidean 3D space, linearly dependent vectors are **parallel**, and linearly independent vectors can be extended into lines that will eventually intersect.

Since we can define a vector space of functions, the following also definition makes sense.

Definition 15.6. We say that two functions f and g are **linearly dependent** if there exist some nonzero constants such that

$$Af(t) + Bg(t) = 0 \tag{15.49}$$

for all values of t. If no such constants exists then we say that f and g are **linearly independent**.

Lesson 16

Linear Equations With Constant Coefficients

Definition 16.1. The general **second order linear equation with constant coefficients** is

$$ay'' + by' + cy = f(t) \tag{16.1}$$

where a, b, and c are constants, and $a \neq 0$ (otherwise (16.1) reduces to a linear first order equation, which we have already covered), and $f(t)$ depends only on t and not on y.

Definition 16.2. The **Linear Differential Operator** corresponding to equation (16.1)

$$L = aD^2 + bD + c \tag{16.2}$$

where

$$D = \frac{d}{dt} \text{ and } D^2 = \frac{d^2}{dt^2} \tag{16.3}$$

is the same operator we introduced in example (15.10). We can also write L as

$$L = a\frac{d^2}{dt^2} + b\frac{d}{dt} + c \tag{16.4}$$

In terms of the operator L, equation (16.1) becomes

$$Ly = f(t) \tag{16.5}$$

Example 16.1. Show that L is a linear operator.

Recall from definition (15.3) we need to show the following to demonstrate that L is linear

$$L(\alpha y + \beta z) = \alpha Ly + \beta Lz \tag{16.6}$$

where α and β are constants and $y(t)$ and $z(t)$ are functions. But

$$\begin{aligned}
L(\alpha y + \beta z) &= (aD^2 + bD + c)(\alpha y + \beta z) && (16.7)\\
&= aD^2(\alpha y + \beta z) + bD(\alpha y + \beta z) + c(\alpha y + \beta z) && (16.8)\\
&= a(\alpha y + \beta z)'' + b(\alpha y + \beta z)' + c(\alpha y + \beta z) && (16.9)\\
&= a(\alpha y'' + \beta z'') + b(\alpha y' + \beta z') + c(\alpha y + \beta z) && (16.10)\\
&= \alpha(ay'' + by' + cy) + \beta(az'' + bz' + cz) && (16.11)\\
&= \alpha Ly + \beta Lz \quad \square && (16.12)
\end{aligned}$$

Definition 16.3. The **characteristic polynomial** corresponding to characteristic polynomial (16.1) is

$$ar^2 + br + c = 0 \tag{16.13}$$

Equation (16.13) is also called the **characteristic equation** of the differential equation.

The following example illustrates why the characteristic equation is useful.

Example 16.2. For what values of r does $y = e^{rt}$ satisfy the differential equation

$$y'' - 4y' + 3y = 0 \tag{16.14}$$

Differentiating $y = e^{rt}$,

$$y' = re^{rt} \tag{16.15}$$

$$y'' = r^2e^{rt} \tag{16.16}$$

Plugging both expressions into (16.14),

$$r^2e^{rt} - 4re^{rt} + 3e^rt = 0 \tag{16.17}$$

Since e^{rt} can never equal zero we can cancel it out of every term,

$$r^2 - 4r + 3 = 0 \tag{16.18}$$

Equation (16.18) is the characteristic equation of (16.14). Factoring it,

$$(r-3)(r-1) = 0 \tag{16.19}$$

Hence both $r = 1$ and $r = 3$. This tells us each of the following functions are solutions of (16.14):

$$y = e^t \tag{16.20}$$
$$y = e^{3t} \tag{16.21}$$

We will see shortly how to combine these to get a more general solution. □

We can generalize the last example as follows.

Theorem 16.4. If r is a root of the characteristic equation of $Ly = 0$, then e^{rt} is a solution of $Ly = 0$.

Proof.

$$L[e^{rt}] = a(e^{rt})'' + b(e^{rt})' + c(e^{rt}) \tag{16.22}$$
$$= ar^2e^{rt} + bre^{rt} + ce^{rt} \tag{16.23}$$
$$= (ar^2 + br + c)e^{rt} \tag{16.24}$$

Since r is a root of the characteristic equation,

$$ar^2 + br + c = 0 \tag{16.25}$$

Hence

$$L[e^{rt}] = 0 \tag{16.26}$$

Thus $y = e^{rt}$ is a solution of $Ly = 0$. □

Theorem 16.5. If the characteristic polynomial has a repeated root r, then $y = te^{rt}$ is a solution of $Ly = 0$.

Proof.

$$L(te^{rt}) = a(te^{rt})'' + b(te^{rt})' + c(te^{rt}) \tag{16.27}$$
$$= a(e^{rt} + rte^{rt})' + b(e^{rt} + rte^{rt}) + cte^{rt} \tag{16.28}$$
$$= a(2re^{rt} + r^2te^{rt}) + b(e^{rt} + rte^{rt}) + cte^{rt} \tag{16.29}$$
$$= e^{rt}(2ar + b + (ar^2 + br + c)t) \tag{16.30}$$

Since r is a root, $r^2 + br + c = 0$. Hence

$$L(te^{rt}) = e^{rt}(2ar + b) \tag{16.31}$$

Since r is root, from the quadratic equation,

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{16.32}$$

When a root is repeated, the square root is zero, hence

$$r = -\frac{b}{2a} \tag{16.33}$$

Rearranging gives

$$2ar + b = 0 \tag{16.34}$$

whenever r is a repeated root. Substituting equation (16.34) into equation (16.31) gives $L(te^{rt}) = 0$. □

Example 16.3. We can use theorem (16.4) to find two solutions of the homogeneous linear differential equation

$$y'' - 7y' + 12y = 0 \tag{16.35}$$

The characteristic equation is

$$r^2 - 7r + 12 = 0 \tag{16.36}$$

Factoring gives

$$(r - 3)(r - 4) = 0 \tag{16.37}$$

Since the roots are $r = 3$ and $r = 4$, two solutions of the differential equation (16.35) are

$$y_{H1} = e^{3t} \tag{16.38}$$

$$y_{H2} = e^{4t} \tag{16.39}$$

Thus for any real numbers A and B,

$$y = Ae^{3t} + Be^{4t} \tag{16.40}$$

is also a solution. □

Because equation (16.14) involves a second-order derivative the solution will in general include two constants of integration rather than the single arbitrary constant that we had when we were solving first order equations. These **initial conditions** are expressed as the values of both the function and its derivative at the same point, e.g.,

$$\left.\begin{aligned} y(t_0) &= y_0 \\ y'(t_0) &= y_1 \end{aligned}\right\} \tag{16.41}$$

The **second order linear initial value problem** is then

$$\left.\begin{aligned} Ly &= 0 \\ y(t_0) &= y_0 \\ y'(t_0) &= y_1 \end{aligned}\right\} \tag{16.42}$$

This is not the only way to express the constraints upon the solution. It is also possible to have **boundary conditions**, of which several types are possible:

$$y(t_0) = y_0, y(t_1) = y_1 \qquad \text{(Dirichlet Boundary Conditions)} \tag{16.43}$$

$$y(t_0) = y_0, y'(t_1) = y_1 \qquad \text{(Mixed Boundary Condition)} \tag{16.44}$$

$$y'(t_0) = y_0, y'(t_1) = y_1 \qquad \text{(Neumann Boundary Conditions)} \tag{16.45}$$

Differential equations combined with boundary conditions are called **Boundary Value Problems**. Boundary Value Problems are considerably more complex than Initial Value Problems and we will not study them in this class.

Definition 16.6. The **homogeneous linear second order differential equation with constant coefficients** is written as

$$ay'' + by' + cy = 0 \tag{16.46}$$

or

$$Ly = 0 \tag{16.47}$$

We will denote a solution to the homogeneous equations as $y_H(t)$ to distinguish it from a solution of

$$ay'' + by' + cy = f(t) \tag{16.48}$$

If there are multiple solutions to the homogeneous equation we will number them $y_{H1}, y_{H2}, \ldots$. We will call any solution of (16.48) a **particular solution** and denote it as $y_P(t)$. If there are multiple particular solutions we will also number them if we need to.

Theorem 16.7. If $y_H(t)$ is a solution to

$$ay'' + by' + cy = 0 \tag{16.49}$$

and $y_P(t)$ is a solution to

$$ay'' + by' + cy = f(t) \tag{16.50}$$

then

$$y = y_H(t) + y_P(t) \tag{16.51}$$

is also a solution to (16.50).

Proof. We are give $Ly_H = 0$ and $Ly_P = f(t)$. Hence

$$Ly = L(y_H + y_P) = Ly_H + Ly_P = 0 + f(t) = f(t) \tag{16.52}$$

Hence $y = h_H + y_P$ is a solution. □

General Principal. The general solution to

$$Ly = ay'' + by' + cy = f(t) \tag{16.53}$$

is the sum of a homogeneous and a particular part:

$$y = y_H(t) + y_P(t) \tag{16.54}$$

where $Ly_H = 0$ and $Ly_P = f(t)$.

Theorem 16.8. Principle of Superposition If $y_{H1}(t)$ and $y_{H2}(t)$ are both solutions of $Ly = 0$, then any linear combination

$$y_H(t) = Ay_{H1}(t) + By_{H2}(t) \tag{16.55}$$

is also a solution of$Ly = 0$.

Proof. Since $y_{H1}(t)$ and $y_{H2}(t)$ are solutions,

$$Ly_{H1} = 0 = Ly_{H2} \tag{16.56}$$

Since L is a linear operator,

$$\begin{aligned} Ly_H &= L[Ay_{H1} + By_{H2}] && (16.57)\\ &= ALy_{H1} + BLy_{H2} && (16.58)\\ &= 0 && (16.59) \end{aligned}$$

Hence any linear combination of solutions to the homogeneous equation is also a solution of the homogeneous equation. □

General Solution of the Homogeneous Equation with Constant Coefficients. From theorem (16.4) we know that e^{rt} is a solution of $Ly = 0$ whenever r is a root of the characteristic equation. If r is a repeated root, we also know from theorem (16.5) that te^{rt} is also a solution. Thus we can always find two solutions to the homogeneous equation with constant coefficients by finding the roots of the characteristic equation. In general these are sufficient to specify the complete solution.

The general solution to

$$ay'' + by' + cy = 0 \tag{16.60}$$

is given by

$$y = \begin{cases} Ae^{r_1 t} + Be^{r_2 t} & r_1 \neq r_2 \text{ (distinct roots)} \\ (A + Bt)e^{rt} & r = r_1 = r_2 \text{ (repeated root)} \end{cases} \tag{16.61}$$

where r_1 and r_2 are roots of the characteristic equation $ar^2 + br + c = 0$.

Example 16.4. Solve the initial value problem

$$\left.\begin{aligned} y'' - 6y' + 8y = 0 \\ y(0) = 1 \\ y'(0) = 1 \end{aligned}\right\} \tag{16.62}$$

The characteristic equation is

$$0 = r^2 - 6r + 8 = (r - 4)(r - 2) \tag{16.63}$$

The roots are $r = 2$ and $r = 4$, hence

$$y = Ae^{2t} + Be^{4t} \tag{16.64}$$

From the first initial condition,

$$1 = A + B \tag{16.65}$$

Differentiating (16.64)

$$y' = 2Ae^{2t} + 4Be^{4t} \tag{16.66}$$

From the second initial condition

$$1 = 2A + 4B \tag{16.67}$$

From (16.65), $B = 1 - A$, hence

$$1 = 2A + 4(1 - A) = 4 - 2A \implies 2A = 3 \implies A = \frac{3}{2} \tag{16.68}$$

hence

$$B = 1 - A = 1 - \frac{3}{2} = -\frac{1}{2} \tag{16.69}$$

The solution to the IVP is

$$y = \frac{3}{2}e^{2t} - \frac{1}{2}e^{4t} \quad \square \tag{16.70}$$

Example 16.5. Solve the initial value problem

$$\left.\begin{aligned} y'' - 6y' + 9y &= 0 \\ y(0) &= 4 \\ y'(0) &= 17 \end{aligned}\right\} \tag{16.71}$$

The characteristic equation is

$$0 = r^2 - 6r + 9 = (r-3)^2 \tag{16.72}$$

Since there is a repeated root $r = 3$, the general solution of the homogeneous equation is

$$y = Ae^{3t} + Bte^{3t} \tag{16.73}$$

By the first initial condition, we have $4 = y(0) = A$.Hence

$$y = 4e^{3t} + Bte^{3t} \tag{16.74}$$

Differentiating,

$$y' = 12e^{3t} + Be^{et} + 3Bte^{3t} \tag{16.75}$$

From the second initial condition,

$$17 = y'(0) = 12 + B + 3B = 12 + B \implies B = 5 \tag{16.76}$$

Hence the solution of the initial value problem is

$$y = (4 + 5t)e^{3t}. \quad \square \tag{16.77}$$

Lesson 17

Some Special Substitutions

Equations with no y dependence.

If $c = 0$ in L then the differential equation

$$Ly = f(t) \tag{17.1}$$

simplifies to

$$ay'' + by' = f(t) \tag{17.2}$$

In this case it is possible to solve the equation by making the change of variables $z = y'$, which reduces the ODE to a first order linear equation in z. This works even when a or b have t dependence. This method is illustrated in the following example.

Example 17.1. Find the general solution of the homogeneous linear equation

$$y'' + 6y' = 0 \tag{17.3}$$

Making the substitution $z = y'$ in (17.3) gives us

$$z' + 6z = 0 \tag{17.4}$$

Separating variables and integrating gives

$$\int \frac{dz}{z} = -\int 6dt \implies z = Ce^{-6t} \tag{17.5}$$

Replacing z with its original value of $z = y'$,

$$\frac{dy}{dt} = Ce^{-6t} \tag{17.6}$$

Hence

$$\begin{aligned} y &= \int \frac{dy}{dt} dt && (17.7)\\ &= \int Ce^{-6t} dt \qquad = && (17.8)\\ &= -\frac{1}{6}Ce^{-6t} + C' && (17.9) \end{aligned}$$

Since $-C/6$ is still a constant we can rename it $C'' = -C/6$, then rename C'' back to C, giving us

$$y = Ce^{-6t} + C' \quad \square \tag{17.10}$$

Example 17.2. Find the general solution o

$$y'' + 6y' = t \tag{17.11}$$

We already have solved the homogeneous problem $y'' + 6y' = 0$ in example (17.1). From that we expect that

$$y = Y_P + Y_H \tag{17.12}$$

where

$$Y_H = Ce^{-6t} + C' \tag{17.13}$$

To see what we get if we substitute $z = y'$ into (17.11) and obtain the first order linear equation

$$z' + 6z = t \tag{17.14}$$

An integrating factor is $\mu = \exp\left(\int 6dt\right) = e^{6t}$, so that

$$\frac{d}{dt}\left(ze^{6t}\right) = (z' + 6t)\, e^{6t} = te^{6t} \tag{17.15}$$

Integrating,

$$\begin{aligned} \int \frac{d}{dt}\left(ze^{6t}\right) dt &= \int te^{6t} dt && (17.16)\\ ze^{6t} &= \frac{1}{36}(6t-1)e^{6t} + C && (17.17)\\ z &= \frac{1}{6}t - \frac{1}{36} + Ce^{-6t} && (17.18) \end{aligned}$$

Therefore since $z = y'$,

$$\frac{dy}{dt} = \frac{1}{6}t - \frac{1}{36} + Ce^{-6t} \tag{17.19}$$

Integrating gives

$$y = \frac{1}{12}t^2 - \frac{1}{36}t - \frac{1}{6}Ce^{-6t} + C' \tag{17.20}$$

It is customary to combine the $(-1/6)C$ into a single unknown constant, which we again name C,

$$y = \frac{1}{12}t^2 - \frac{1}{36}t + Ce^{-6t} + C' \tag{17.21}$$

Comparing this with (17.12) and (17.13) we see that a particular solution is

$$y_P = \frac{1}{12}t^2 - \frac{1}{36} \quad \square \tag{17.22}$$

This method also works when $b(t)$ has t dependence.

Example 17.3. Solve

$$\left.\begin{aligned} y'' + 2ty' &= t \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}\right\} \tag{17.23}$$

Substituting $z = y'$ gives

$$z' + 2tz = t \tag{17.24}$$

An integrating factor is $\exp\left(\int 2tdt\right) = e^{t^2}$. Hence

$$\frac{d}{dt}\left(ze^{t^2}\right) = te^{t^2} \tag{17.25}$$

$$ze^{t^2} = \int te^{t^2}dt + C = \frac{1}{2}e^{t^2} + C \tag{17.26}$$

$$z = \frac{1}{2} + Ce^{-t^2} \tag{17.27}$$

From the initial condition $y'(0) = z(0) = 0$, $C = -1/2$, hence

$$\frac{dy}{dt} = \frac{1}{2} - \frac{1}{2}e^{-t^2} \tag{17.28}$$

Changing the variable from t to s and integrating from $t = 0$ to t gives

$$\int_0^t \frac{dy}{ds}ds = \int_0^t \frac{1}{2}ds - \frac{1}{2}\int_0^t e^{-s^2}ds \tag{17.29}$$

From (4.91),

$$\int_0^t e^{-s^2} ds = \frac{\sqrt{\pi}}{2}\operatorname{erf}(t) \tag{17.30}$$

hence

$$y(t) - y(0) = \frac{t}{2} - \frac{\sqrt{\pi}}{4}\operatorname{erf}(t) \tag{17.31}$$

From the initial condition $y(0) = 1$,

$$y(t) = 1 + \frac{t}{2} - \frac{\sqrt{\pi}}{4}\operatorname{erf}(t) \quad \Box \tag{17.32}$$

] This method also works for nonlinear equations.

Example 17.4. Solve $y'' + t(y')^2 = 0$

Let $z = y'$, then $z' = y''$, hence $z' + tz^2 = 0$. Separating variables,

$$\frac{dz}{dt} = -tz^2 \tag{17.33}$$

$$z^{-2}dz = -tdt \tag{17.34}$$

$$\frac{1}{z} + C_1 = -\frac{1}{2}t^2 \tag{17.35}$$

Solving for z,

$$\frac{dy}{dt} = z = \frac{-1}{t^2/2 + C_1} = \frac{-2}{t^2 + C_2} \tag{17.36}$$

where $C_2 = 2C_1$ hence

$$y = -2\arctan\frac{t}{k} + C \tag{17.37}$$

where $k = \sqrt{C_2}$ and C are arbitrary constants of integration. $\Box$

Equations with no t dependence

If there is no t-dependence in a linear ODE we have

$$Ly = ay'' + by' + cy = 0 \tag{17.38}$$

This is a homogeneous differential equation with constant coefficients and reduces to a case already solved.

If the equation is nonlinear we can make the same substitution

Example 17.5. Solve $yy'' + (y')^2 = 0$.

Making the substitution $z = y'$ and $y'' = z'$ gives

$$zz' + z^2 = 0 \tag{17.39}$$

We can factor out a z,

$$z(z' + z) = 0 \tag{17.40}$$

hence either $z = 0$ or $z' = -z$. The first choice gives

$$\frac{dy}{dt} = 0 \implies y_1 = C \tag{17.41}$$

as a possible solution. The second choice gives

$$\frac{dz}{z} = -dt \implies \ln z = -t + k \implies z = Ke^{-t} \tag{17.42}$$

where $K = e^{-k}$ is a constant. Hence

$$\frac{dy}{dt} = Ke^{-t} \implies dy = Ke^{-t}dt \tag{17.43}$$

$$y = -Ke^{-t} + K_1 \tag{17.44}$$

where K_1 is a second constant of integration. If we let $K_0 = -K$ then this solution becomes

$$y_2 = K_0e^{-t} + K_1 \tag{17.45}$$

Since we cannot distinguish between the two arbitrary constants K_1 in the second solution and C in the first, we see that the first solution is actually found as part of the second solution. Hence (17.45) gives the most general solution. $\square$

Factoring a Linear ODE

The D operator introduced in example (15.10) will be quite useful in studying higher order linear differential equations. We will usually write it as a symbol to the left of a function, as in

$$Dy = \frac{dy}{dt} \tag{17.46}$$

where D is interpreted as an operator, i.e., D does something to whatever is written to the right of it. The proper analogy is like a matrix: think of D as an $n \times n$ matrix and y as a column vector of length n (or an $n \times 1$

matrix). Then we are allowed to multiply D by y on the right, to gives us another function. Like the matrix an vector, we are not allowed to reverse the order of the D and whatever it operates on. Some authors write $D[y]$, $D(y)$, $D_t y$, or $\partial_t y$ instead of Dy.

Before we begin our study of higher order equations, we will look at what the D operator represents in terms of linear first order equations. While it doesn't really add anything to our understanding of linear first order equations, looking at how it can be used to describe these equations will help us to understand its use in higher order linear equations.

We begin by rewriting the first order linear differential equation

$$\frac{dy}{dt} + p(t)y = q(t) \tag{17.47}$$

as

$$Dy + p(t)y = q(t) \tag{17.48}$$

The trick here is to think like matrix multiplication: we are still allowed the distributive law, so we can factor out the y on the left hand side, but only on the right. In other words, we can say that

$$[D + p(t)]y = q(t) \tag{17.49}$$

Note that we **cannot** factor out the y on the left, because

$$Dy \neq yD \tag{17.50}$$

so it would be **incorrect to say anything like**

$$y(D + p(t)) = q(t) \tag{17.51}$$

In fact, anytime you see a D that is not multiplying something on its right, that should ring a bell telling you that something is wrong and you have made a calculation error some place.

Continuing with equation 17.49 we can now reformulate the general initial value problem as

$$\left.\begin{aligned} [D + p(t)]y &= q(t) \\ y(t_0) &= y_0 \end{aligned}\right\} \tag{17.52}$$

The D operator has some useful properites. In fact, thinking in terms of matrices, it would be nice if we could find an expression for the inverse of D so that we could solve for y. If M is a matrix then its inverse M^{-1} has the property that

$$MM^{-1} = M^{-1}M = I \tag{17.53}$$

where I is the identity matrix.

Since the inverse of the derivative is the integral, then for any function $f(t)$,

$$D^{-1}f(t) = \int f(t)dt + C \tag{17.54}$$

Hence

$$DD^{-1}f(t) = D\left(\int f(t) + C\right) = f(t) \tag{17.55}$$

and

$$D^{-1}Df(t) = \int Df(t)dt + C = f(t) + C \tag{17.56}$$

We write the general linear first order initial value problem in terms of the linear differential operator $D = d/dt$ as

$$[D + p(t)]\, y = q(t), y(t_0) = y_0 \tag{17.57}$$

From the basic properties of derivatives,

$$De^{f(t)} = \frac{d}{dt}e^{f(t)} = e^{f(t)}\frac{d}{dt}f(t) = e^{f(t)}Df(t) \tag{17.58}$$

From the product rule,

$$Df(t)g(g) = f(t)Dg(t) + g(t)Df(t) \tag{17.59}$$

Hence

$$D(e^{f(t)}y) = e^{f(t)}Dy + yDe^{f(t)} \tag{17.60}$$

$$= e^{f(t)}Dy + ye^{f(t)}Df(t) \tag{17.61}$$

$$= e^{f(t)}\left(Dy + yDf(t)\right) \tag{17.62}$$

If we let

$$f(t) = \int p(t)dt = D^{-1}p(t)dt \tag{17.63}$$

then (using our previous definition of μ),

$$e^{f(t)} = \exp\left(\int p(t)dt\right) = \mu(t) \tag{17.64}$$

hence

$$D(e^{f(t)}y) = e^{f(t)}(Dy + yDD^{-1}p(t)) \tag{17.65}$$
$$= e^{f(t)}(Dy + yp(t)) \tag{17.66}$$
$$= e^{f(t)}(D + p(t))y \tag{17.67}$$
$$D(\mu(t)y) = \mu(t)(D + p(t))y \tag{17.68}$$
$$= \mu(t)q(t) \tag{17.69}$$

Applying D^{-1} to both sides,

$$D^{-1}D(\mu(t)y) = D^{-1}(\mu(t)q(t)) \tag{17.70}$$

The D^{-1} and D are only allowed to annihilate one another if we add a constant of integration (equation 17.56)

$$\mu(t)y = D^{-1}(\mu(t)q(t)) + C \tag{17.71}$$

or

$$y = \frac{1}{\mu}\left(D^{-1}\mu q + C\right) \tag{17.72}$$

which is the same result we had before (see equation 4.18):

$$y = \frac{1}{\mu(t)}\left(\int \mu(t)q(t)dt + C\right) \tag{17.73}$$

This formalism hasn't really given us anything new for a general linear equation yet. When the function p is a constant, however, it does give us a useful way to look at things.

Suppose that

$$p(t) = A \tag{17.74}$$

where A is an constant. Then the differential equation is

$$(D + A)y = q(t) \tag{17.75}$$

This will become useful when we solve higher order linear equations with constant coefficients because they can be factored:

$$(D + A)(D + B)y = D^2y + (A + B)Dy + ABy \tag{17.76}$$

Thus any equation of the form

$$y'' + ay' + by = q(t) \tag{17.77}$$

can be rewritten by solving

$$a = A + B \tag{17.78}$$

$$b = AB \tag{17.79}$$

for A and B. Then the second order equation is reduced to a sequence of first order equations

$$(D + A)z = q(t) \tag{17.80}$$

$$(D + B)y = z(t) \tag{17.81}$$

One first solves the first equation for z then plugs the solution into the second equation to solve for y.

Lesson 18

Complex Roots

We know form algebra that the roots of the characteristic equation

$$ar^2 + br + c = 0 \tag{18.1}$$

are given by the quadratic formula

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{18.2}$$

When

$$b^2 < 4ac \tag{18.3}$$

the number in the square root will be negative and the roots will be complex.

Definition 18.1. A **complex number** is a number

$$z = a + bi \tag{18.4}$$

where a, b are real numbers (possibly zero) and

$$i = \sqrt{-1} \tag{18.5}$$

To find the square root of a negative number we factor out the -1 and use $i = \sqrt{-1}$, and use the result that

$$\sqrt{-a} = \sqrt{(-1)(a)} = \sqrt{-1}\sqrt{a} = i\sqrt{a} \tag{18.6}$$

Example 18.1. Find $\sqrt{-9}$.

$$\sqrt{-9} = \sqrt{(-1)(9)} = \sqrt{-1}\sqrt{9} = 3i \quad \square \tag{18.7}$$

Example 18.2. Find the roots of

$$r^2 + r + 1 = 0 \tag{18.8}$$

We have $a = b = c = 1$ hence according to (18.2) the roots are

$$r = \frac{-1 \pm \sqrt{1^2 - (4)(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2} \tag{18.9}$$

Since $-3 < 0$ it does not have a real square root;

$$\sqrt{-3} = \sqrt{(3)(-1)} = \sqrt{-1}\sqrt{3} = i\sqrt{3} \tag{18.10}$$

hence

$$r = \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \tag{18.11}$$

Properties of Complex Numbers

1. If $z = a + ib$, where $a, b \in \mathbb{R}$, then we say that a is the **real part** of z and b is the **imaginary part**, and we write

$$\left.\begin{aligned} \operatorname{Re}(z) = \operatorname{Re}(a + ib) = a \\ \operatorname{Im}(z) = \operatorname{Im}(a + ib) = b \end{aligned}\right\} \tag{18.12}$$

2. The **absolute value** of $z = x + iy$ is the distance in the xy plane from the origin to the point (x, y). Hence

$$|x + iy| = \sqrt{x^2 + y^2} \tag{18.13}$$

3. The **complex conjugate** of $z = x + iy$ is a complex number with all of the i's replaced by $-i$, and is denoted by $\overline{z}$,

$$z = x + iy \implies \overline{z} = \overline{x + iy} = x - iy \tag{18.14}$$

4. If $z = x + iy$ is any complex number then

$$|z|^2 = z\overline{z} \tag{18.15}$$

because

$$x^2 + y^2 = (x + iy)(x - iy) \tag{18.16}$$

5. The **phase** of a complex number $z = x + iy$, denoted by Phase(z) is the angle between the $x - axis$ and the line from the origin to the point (x, y)

$$\operatorname{Phase}(z) = \arctan\frac{y}{x} \tag{18.17}$$

Theorem 18.2. Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{18.18}$$

Proof. Use the fact that

$$\left.\begin{aligned} i^2 &= -1 \\ i^3 &= i(i^2) = -i \\ i^4 &= i(i^3) = i(-i) = 1 \\ i^5 &= i(i^4) = i \end{aligned}\right\} \implies \left\{\begin{aligned} i^{4k+1} &= i \\ i^{4k+2} &= -1 \\ i^{4k+3} &= -i \\ i^{4k+4} &= 1 \end{aligned}\right\} \text{ for all } k = 0, 1, 2, \dots \tag{18.19}$$

in the formula's for a Taylor Series of $e^{i\theta}$:

$$e^{i\theta} = 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \tag{18.20}$$

$$= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \cdots \tag{18.21}$$

$$= 1 + i\theta - \frac{\theta^2}{2!} + -i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots \tag{18.22}$$

$$= \overbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right)}^{\text{even powers of } \theta} + i\underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)}_{\text{odd powers of } \theta} \tag{18.23}$$

$$= \cos\theta + i\sin\theta \tag{18.24}$$

where the last step follows because we have used the Taylor series for $\sin\theta$ and $\cos\theta$. □

Theorem 18.3. If $z = a + ib$ where a and b are real numbers than

$$e^z = e^{a+ib} = e^a(\cos\theta + i\sin\theta) \tag{18.25}$$

Theorem 18.4. If $z = x + iy$ where x, y are real, then

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta} \tag{18.26}$$

where $\theta = \text{Phase}(z)$ and $r = |z|$.

Proof. By definition $\theta = \text{Phase}(z)$ is the angle between the x axis and the

line from the origin to the point (x, y). Hence

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{|z|} \tag{18.27}$$

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{|z|} \tag{18.28}$$

Therefore

$$\begin{aligned} z &= x + iy && (18.29)\\ &= |z|\left(\frac{x}{|z|} + i\frac{y}{|z|}\right) && (18.30)\\ &= |z|(\cos\theta + i\sin\theta) && (18.31)\\ &= |z|e^{i\theta} && (18.32) \end{aligned}$$

This form is called the **polar form** of the complex number. □

Roots of Polynomials

1. If $z = x + iy$ is the root of a polynomial, then $\overline{z} = x - iy$ is also a root.
2. Every polynomial of order n has precisely n complex roots.
3. Every odd-ordered polynomial of order n has at least one real root: the number of real roots is either 1, or 3, or 5, ..., or n; the remaining roots are complex conjugate pairs.
4. An even-ordered polynomial of order n has either zero, or 2, or 4, or 6, or ... n real real roots; all of the remaining roots are complex conjugate pairs.

Theorem 18.5. Every complex number z has precisely n unique n^{th} roots.

Proof. Write z in polar form.

$$z = re^{i\theta} = re^{i\theta+2k\pi} = re^{i\theta}e^{2k\pi},\ k = 0, 1, 2, \ldots \tag{18.33}$$

where r and θ are real numbers. Take the n^{th} root:

$$\begin{aligned} \sqrt[n]{z} &= z^{1/n} && (18.34)\\ &= \left(re^{i\theta+2k\pi}\right)^{1/n} && (18.35)\\ &= r^{1/n}e^{i(\theta+2k\pi)/n} && (18.36)\\ &= \sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right) && (18.37) \end{aligned}$$

For $k = 0, 1, 2, \ldots, n-1$ the right hand side produces unique results. But for $k \geq n$, the results start to repeat: $k = n$ gives the same angle as $k = 0$; $k = n+1$ gives the same angle as $k = 1$; and so on. Hence there are precisely n unique numbers. □

Example 18.3. Find the three cube roots of 27.

To find the cube roots we repeat the proof!

$$27 = 27 + (0)i = 27e^{(i)(0)} = 27e^{(i)(0+2\pi)} = 27e^{2k\pi i} \tag{18.38}$$

$$\sqrt[3]{27} = 27^{1/3}(e^{2k\pi i})^{1/3} \tag{18.39}$$

$$= 3e^{2k\pi i/3} \tag{18.40}$$

For $k = 0$ this gives

$$\sqrt[3]{27} = 3 \tag{18.41}$$

For $k = 1$ this gives

$$\sqrt[3]{27} = 3e^{2\pi i/3} = 3\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -\frac{3}{2} + i\frac{3\sqrt{3}}{2} \tag{18.42}$$

For $k = 2$ this gives

$$\sqrt[3]{27} = 3e^{4\pi i/3} = 3\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = -\frac{3}{2} - i\frac{3\sqrt{3}}{2} \tag{18.43}$$

Using $k = 3$ will give us the first result, and so forth, so these are all the possible answers. □

Theorem 18.6. If the roots of

$$ar^2 + br + c = 0 \tag{18.44}$$

are a complex conjugate pair

$$\left.\begin{aligned} r_1 &= \mu + i\omega \\ r_2 &= \mu - i\omega \end{aligned}\right\} \tag{18.45}$$

where $\mu, \omega \in \mathbb{R}$ and $\omega \neq 0$, (this will occur when $b^2 < 4ac$), then the solution of the homogeneous second order linear ordinary differential equation with constant coefficients

$$Ly = ay'' + by' + cy = 0 \tag{18.46}$$

is given by

$$y_H = C_1e^{r_1t} + C_2e^{r_2t} \tag{18.47}$$

$$= e^{\mu t}\left(A\cos\omega t + B\sin\omega t\right) \tag{18.48}$$

where

$$A = C_1 + C_2 \tag{18.49}$$

$$B = i(C_1 - C_2) \tag{18.50}$$

Proof.

$$y_H = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{18.51}$$

$$= C_1 e^{(\mu+i\omega)t} + C_2 e^{(\mu-i\omega)t} \tag{18.52}$$

$$= e^{\mu t}(C_1 e^{i\omega t} + C_2 e^{-i\omega t}) \tag{18.53}$$

$$= e^{\mu t}\left[C_1(\cos\omega t + i\sin\omega t) + C_2(\cos\omega t - i\sin\omega t)\right] \tag{18.54}$$

$$= e^{\mu t}\left[(C_1 + C_2)\cos\omega t + i(C_1 - C_2)\sin\omega t\right] \tag{18.55}$$

$$= e^{\mu t}\left(A\cos\omega t + B\sin\omega t\right) \tag{18.56}$$

□

Example 18.4. Find the general solution of

$$y'' + 6y' + 25y = 0 \tag{18.57}$$

The characteristic polynomial is

$$r^2 + 6r + 25 = 0 \tag{18.58}$$

and the roots are

$$r = \frac{-6 \pm \sqrt{-64}}{2} = \frac{-6 \pm 8i}{2} = -3 \pm 4i \tag{18.59}$$

Hence the general solution is

$$y = e^{-3t} A\cos 4t + B\sin 4t]. \tag{18.60}$$

for any numbers A and B. □

Example 18.5. Solve the initial value problem

$$\left.\begin{aligned} y'' + 2y' + 2y &= 0 \\ y(0) &= 2 \\ y'(0) &= 4 \end{aligned}\right\} \tag{18.61}$$

The characteristic equation is

$$r^2 + 2y + 2 = 0 \tag{18.62}$$

and its roots are given by

$$r = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(2)}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i \tag{18.63}$$

The solution is then

$$y = e^{-t}(A\cos t + B\sin t) \tag{18.64}$$

The first initial condition gives

$$2 = A \tag{18.65}$$

and thus the solution becomes

$$y = e^{-t}(2\cos t + B\sin t) \tag{18.66}$$

Differentiating this solution

$$y' = -e^{-t}(2\cos t + B\sin t) + e^{-t}(-2\sin t + B\cos t) \tag{18.67}$$

The second initial condition gives us

$$4 = -2 + B \implies B = 6 \tag{18.68}$$

Hence

$$y = e^{-t}(2\cos t - 6\sin t) \tag{18.69}$$

□

Summary. General solution to the homogeneous linear equation with constant coefficients.

The general solution to

$$ay'' + by' + cy = 0 \tag{18.70}$$

is given by

$$y = \begin{cases} Ae^{r_1 t} + Be^{r_2 t} & r_1 \neq r_2 \text{ (distinct real roots)} \\ (A + Bt)e^{rt} & r = r_1 = r_2 \text{ (repeated real root)} \\ e^{\mu t}(A\cos\omega t + B\sin\omega t) & r = \mu \pm i\omega \text{ (complex roots)} \end{cases} \tag{18.71}$$

where r_1 and r_2 are roots of the characteristic equation $ar^2 + br + c = 0$.

Why Do Complex Numbers Work?*

We have not just pulled $i = \sqrt{-1}$ out of a hat by magic; we can actually define the Field of real numbers rigorously using the following definition.

Definition 18.7. Let $a, b \in \mathbb{R}$. Then a **Complex Number** is an ordered pair

$$z = (a, b) \tag{18.72}$$

with the following properties:

1. Complex Addition, defined by

$$z + w = (a + c, b + d) \tag{18.73}$$

2. Complex Multiplication, defined by

$$z \times w = (ac - bd, ad + bc) \tag{18.74}$$

where $z = (a, b)$ and $w = (c, d)$ are complex numbers.

Then we can define the real and imaginary parts of z as the components $\text{Re}z = \text{Re}(a, b) = a$ and $\text{Im}z = \text{Im}(a, b) = b$.

The Real Axis is defined as the set

$$\{z = (x, 0) | x \in \mathbb{R}\} \tag{18.75}$$

and the imaginary axis is the set of complex numbers

$$\{z = (0, y) | y \in \mathbb{R}\} \tag{18.76}$$

We can see that there is a one-to-relationship between the real numbers and the set of complex numbers $(x, 0)$ that we have associated with the real axis, and there is also a one-to-one relationship between the set of all complex numbers and the real plane $\mathbb{R}^2$. We sometimes refer to this plane as the complex plane or $\mathbb{C}$.

To see that equations 18.73 and 18.74 give us the type of arithmetic that we expect from imaginary numbers, suppose that $a, b, c \in \mathbb{R}$ and define scalar multiplication by

$$c(a, b) = (ca, cb) \tag{18.77}$$

To see that this works, let $u = (x, 0)$ be any point on the real axis. Then

$$uz = (x, 0) \times (a, b) = (ax - 0b, bx - 0a) = (ax, bx) = x(a, b) \tag{18.78}$$

The motivation for equation 18.74 is the following. Suppose $z = (0, 1)$. Then by 18.74,

$$z^2 = (0, 1) \times (0, 1) = (-1, 0) \tag{18.79}$$

We use the special symbol i to represent the complex number $i = (0, 1)$. Then we can write any complex number $z = (a, b)$ as

$$z = (a, b) = (a, 0) + (b, 0) = a(1, 0) + b(0, 1) \tag{18.80}$$

Since $i = (0, 1)$ multiplication by $(1, 0)$ is identical to multiplication by 1 we have

$$z = (a, b) = a + bi \tag{18.81}$$

and hence from 18.79

$$i^2 = -1 \tag{18.82}$$

The common notation is to represent complex numbers as $z = a + bi$ where $a, b \in \mathbb{R}$, where i represents the square root of -1. It can easily be shown that the set of complex numbers defined in this way have all of the properties of a Field.

Theorem 18.8. Properties of Complex Numbers

1. Closure. The set of complex numbers is closed under addition and multiplication.

2. Commutivity. For all complex number w, z,

$$\left.\begin{array}{c} w + z = z + w \\ wz = zw \end{array}\right\} \tag{18.83}$$

3. Associativity. For all complex numbers u, v, w,

$$\left.\begin{array}{c} (u + v) + w = u + (v + w) \\ (uv)w = u(vw) \end{array}\right\} \tag{18.84}$$

4. Identities. For all complex numbers z,

$$\left.\begin{array}{c} z + 0 = 0 + z = z \\ z1 = 1z = z \end{array}\right\} \tag{18.85}$$

5. Additive Inverse. For every complex number z, there exists some unique complex number w such that $z + w = 0$. We call $w = -z$ and

$$z + (-z) = (-z) + z = 0 \tag{18.86}$$

6. Multiplicative Inverse. For every nonzero complex number z, there exists some unique complex number w such that $zw = wz = 1$. We write $w = 1/z$, and

$$z(z^{-1}) = (z^{-1})z = 1 \text{ or } z(1/z) = (1/z)z = 1 \tag{18.87}$$

7. Distributivity. For all complex numbers u, w, z,

$$u(w + z) = uw + uz \tag{18.88}$$

Lesson 19

Method of Undetermined Coefficients

We showed in theorem (22.7) that a particular solution for

$$ay'' + by' + cy = f(t) \tag{19.1}$$

is given by

$$y_P = \frac{1}{a}e^{r_2 t}\int e^{(r_1 - r_2)t}\left(\int f(t)e^{-r_1 t}dt\right)dt \tag{19.2}$$

where r_1 and r_2 are roots of the characteristic equation. While this formula will work for any function $f(t)$ it is difficult to memorize and there is sometimes an easier to find a particular solution. In the **method of undetermined coefficients** we do this:

1. Make an educated guess on the form of $y_P(t)$ up to some unknown constant multiple, based on the form of $f(t)$.
2. Plug y_P into (19.1).
3. Solve for unknown coefficients.
4. If there is a solution then you have made a good guess, and are done.

The method is illustrated in the following example.

Example 19.1. Find a solution to

$$y'' - y = t^2 \tag{19.3}$$

Since the characteristic equation is $r^2 - 1 = 0$, with roots $r = \pm 1$, we know that the homogeneous solution is

$$y_H = C_1 e^t + C_2 e^{-t} \tag{19.4}$$

As a guess to the particular solution we try

$$y_P = At^2 + Bt + C \tag{19.5}$$

Differentiating,

$$y'_P = 2At + B \tag{19.6}$$
$$y''_P = 2A \tag{19.7}$$

Substituting into (19.1),

$$2A - At^2 - Bt - C = t^2 \tag{19.8}$$

Since each side of the equation contains polynomials, we can equate the coefficients of the power on each side of the equation. Thus

$$\text{Coefficients of } t^2: \quad -A = 1 \tag{19.9}$$
$$\text{Coefficients of } t: \quad B = 0 \tag{19.10}$$
$$\text{Coefficients of } t^0: \quad 2A - C = 0 \tag{19.11}$$

The first of these gives $A = -1$, the third $C = 2A = -2$. Hence

$$y_P = -t^2 - 2 \tag{19.12}$$

Substitution into (19.1) verifies that this is, in fact, a particular solution. The complete solution is then

$$y = y_H + y_P = C_1 e^r + C_2 e^{-t} - t^2 - 2 \quad \square \tag{19.13}$$

Use of this method depends on being able to come up with a good guess. Fortunately, in the case where f is a combination of polynomials, exponentials, sines and cosines, a good guess is given by the following heuristic.

1. If $f(t)$ is a polynomial of degree n, use

$$y_P = a_n t^n + a_{n-1} t^{n-1} + \cdots a_2 t^2 + a_1 t + a_0 \tag{19.14}$$

2. If $f(t) = e^{rt}$ and r is not a root of the characteristic equation, try

$$y_P = Ae^{rt} \tag{19.15}$$

3. If $f(t) = e^{rt}$ and r is a root of the characteristic equation, but is not a repeated root, try
$$y_P = Ate^{rt} \tag{19.16}$$

4. If $f(t) = e^{rt}$ and r is a repeated root of the characteristic equation, try
$$y_P = At^2e^{rt} \tag{19.17}$$

5. If $f(t) = \alpha \sin \omega t + \beta \cos \omega t$, where $\alpha, \beta, \omega \in \mathbb{R}$, and neither $\sin \omega t$ nor $\cos \omega t$ are solutions of the homogeneous equation, try
$$y_P = A \cos \omega t + B \sin \omega t \tag{19.18}$$
If (19.18) is a solution of the homogeneous equation, instead try
$$y_P = t(A \cos \omega t + B \sin \omega t) \tag{19.19}$$

6. If f is a product of polynomials, exponentials, and/or sines and cosines, use a product of polynomials, exponentials, and/or sines and cosines. If any of the terms in the product is a solution of the homogeneous equation, multiply the entire solution by t or t^2, whichever ensures that no terms in the guess are a solution of $Ly = 0$.

Example 19.2. Solve
$$y'' + y' - 6y = 2t \tag{19.20}$$
The characteristic equation is
$$r^2 + r - 6 = (r - 2)(r + 3) = 0 \tag{19.21}$$
hence
$$y_H = C_1e^{2t} + C_2e^{-3t} \tag{19.22}$$
Since the forcing function (right-hand side of the equation) is $2t$ we try a particular function of
$$y_P = At + B \tag{19.23}$$
Differentiating,
$$y_P' = A \tag{19.24}$$
$$y_P'' = 0 \tag{19.25}$$
Substituting back into the differential equation,
$$0 + A - 6(At + B) = 6t \tag{19.26}$$
$$-6At + A + B = 6t \tag{19.27}$$

Equating like coefficients,

$$-6A = 6 \implies A = -1 \tag{19.28}$$
$$A + B = 0 \implies B = -A = 1 \tag{19.29}$$
$$y_p = -t + 1 \tag{19.30}$$

hence the general solution is

$$y = C_1 e^{2t} + C_2 e^{-3t} - t + 1 \qquad \square \tag{19.31}$$

Example 19.3. Find the general solution to

$$y'' - 3y' - 4y = e^{-t} \tag{19.32}$$

The characteristic equation is

$$r^2 - 3r - 4 = (r - 4)(r + 1) = 0 \tag{19.33}$$

so that

$$y_H = C_1 e^{4t} + C_2 e^{-t} \tag{19.34}$$

From the form of the forcing function we are tempted to try

$$y_P = Ae^{-t} \tag{19.35}$$

but that is already part of the homogeneous solution, so instead we try

$$y_P = Ate^{-t} \tag{19.36}$$

Differentiating,

$$y' = Ae^{-t} - Ate^{-t} \tag{19.37}$$
$$y'' = -2Ae^{-t} + Ate^{-t} \tag{19.38}$$

Substituting into the differential equation,

$$-2Ae^{-t} + Ate^{-t} - 3(Ae^{-t} - Ate^{-t}) - 4(Ate^{-t}) = e^{-t} \tag{19.39}$$
$$-2A + At - 3(A - At) - 4(At) = 1 \tag{19.40}$$
$$-2A + At - 3A + 3At - 4At = 1 \tag{19.41}$$
$$-5A = 1 \tag{19.42}$$
$$A = -\frac{1}{5} \tag{19.43}$$

hence

$$y_P = -\frac{1}{5}te^{-t} \tag{19.44}$$

and the general solution of the differential equation is

$$y = C_1 e^{4t} + C_2 e^{-t} - \frac{1}{5}te^{-t} \qquad \square \tag{19.45}$$

Example 19.4. Solve

$$\left.\begin{aligned} y'' + 4y &= 3\sin 2t \\ y(0) &= 0 \\ y'(0) &= -1 \end{aligned}\right\} \tag{19.46}$$

The characteristic equation is

$$r^2 + 4 = 0 \tag{19.47}$$

So the roots are $\pm 2i$ and the homogeneous solution is

$$y_H = C_1 \cos 2t + C_2 \sin 2t \tag{19.48}$$

For a particular solution we use

$$y = t(A\cos 2t + B\sin 2t) \tag{19.49}$$

$$y' = Aa\cos 2t + B\sin 2t + t(2B\cos 2t - 2A\sin 2t) \tag{19.50}$$

$$y'' = 4B\cos 2t - 4A\sin 2t + t(-4A\cos 2t - 4B\sin 2t) \tag{19.51}$$

Plugging into the differential equation,

$$\begin{aligned} 4B\cos 2t - 4A\sin 2t + t(-4A\cos 2t - 4B\sin 2t)& \\ + 4t(A\cos 2t + B\sin 2t) &= 3\sin 2t \end{aligned} \tag{19.52}$$

Canceling like terms,

$$4B\cos 2t - 4A\sin 2t = 3\sin 2t \tag{19.53}$$

equating coefficients of like trigonometric functions, $A = 0$, $B = \frac{3}{4}$, hence

$$y = C_1\cos 2t + C_2\sin 2t + \frac{3}{4}t\sin 2t \tag{19.54}$$

From the first initial condition, $y(0) = 0$,

$$0 = C_1 \tag{19.55}$$

hence

$$y = C_2\sin 2t + \frac{3}{4}t\sin 2t \tag{19.56}$$

Differentiating,

$$y' = 2C_2\cos 2t + \frac{3}{4}\sin 2t + \frac{3}{4}t\cos 2t \tag{19.57}$$

The second initial condition is $y'(0) = -1$ so that

$$-1 = 2C_2 \tag{19.58}$$

hence

$$y = -\frac{1}{2}\sin 2t + \frac{3}{4}t\sin 2t \quad \square \tag{19.59}$$

Example 19.5. Solve the initial value problem

$$\left.\begin{array}{r} y'' - y = t + e^{2t} \\ y(0) = 0 \\ y'(0) = 1 \end{array}\right\} \tag{19.60}$$

The characteristic equation is $r^2 - 1 = 0$ so that

$$y_H = C_1e^t + C_2e^{-t} \tag{19.61}$$

For a particular solution we try a linear combination of particular solutions for each of the two forcing functions,

$$y_P = At + B + Ce^{2t} \tag{19.62}$$

$$y_P' = A + 2Ce^{2t} \tag{19.63}$$

$$y_P'' = 4Ce^{2t} \tag{19.64}$$

Substituting into the differential equation,

$$4Ce^{2t} - At - B - Ce^{2t} = t + e^{2t} \tag{19.65}$$

$$3Ce^{2t} - At - B = t + e^{2t} \tag{19.66}$$

$$A = -1 \tag{19.67}$$

$$B = 0 \tag{19.68}$$

$$C = \frac{1}{3} \tag{19.69}$$

Hence

$$y = C_1e^t + C_2e^{-t} - t + \frac{1}{3}e^{2t} \tag{19.70}$$

From the first initial condition, $y(0) = 0$,

$$0 = C_1 + C_2 + \frac{1}{3} \tag{19.71}$$

From the second initial condition $y'(0) = 1$,

$$1 = C_1 - C_2 + \frac{2}{3} \tag{19.72}$$

Adding the two equations gives us $C_1 = 0$ and substitution back into either equation gives $C_2 = -1/3$. Hence

$$y = -\frac{1}{3}e^{-t} - t + \frac{1}{3}e^{2t} \qquad \square \tag{19.73}$$

Here are some typical guesses for the particular solution that will work for common types of forcing functions.

Forcing Function	Particular Solution
Constant	A
t	$At+B$
at^2+bt+c	At^2+Bt+C
$a_nt^n+\cdots+a_0$	$A_nt^n+\cdots+A_0$
$a\cos\omega t$	$A\cos\omega t+B\sin\omega t$
$b\sin\omega t$	$A\cos\omega t+B\sin\omega t$
$t\cos\omega t$	$(At+B)\cos\omega t+(Ct+D)\sin\omega t$
$t\sin\omega t$	$(At+B)\cos\omega t+(Ct+D)\sin\omega t$
$(a_nt^n+\cdots+a_0)\sin\omega t$	$(A_nt^n+\cdots+A_0)\cos\omega t+$ $(A_nt^n+\cdots+A_0)\sin\omega t$
$(a_nt^n+\cdots+a_0)\cos\omega t$	$(A_nt^n+\cdots+A_0)\cos\omega t+$ $(A_nt^n+\cdots+A_0)\sin\omega t$
e^{at}	e^at
te^{at}	$(At+B)e^at$
$t\sin\omega te^{at}$	$e^at((At+B)\sin\omega t+(Ct+D)\cos\omega t)$
$(a_nt^n+\cdots+a_0)e^{at}$	$(A_nt^n+\cdots+A_0)e^{at}$
$(a_nt^n+\cdots+a_0)e^{at}\cos\omega t$	$(A_nt^n+\cdots+A_0)e^{at}\cos\omega t+$ $(A_nt^n+\cdots+A_0)e^{at}\sin\omega t$

If the particular solution shown is already part of the homogeneous solution you should multiply by factors of t until it no longer is a term in the homogeneous solution.

Lesson 20

The Wronskian

We have seen that the sum of any two solutions y_1, y_2 to

$$ay'' + by' + cy = 0 \tag{20.1}$$

is also a solution, so a natural question becomes the following: how many different solutions do we need to find to be certain that we have a general solution? The answer is that **every solution of (20.1) is a linear combination of two linear independent solutions.** In other words, if y_1 and y_2 are **linearly independent** (see definition (15.6)), i.e, there is no possible combination of constants A and B, both nonzero, such that

$$Ay_1(t) + By_2(t) = 0 \tag{20.2}$$

for all t, and if both y_1 and y_2 are solutions, then every solution of (20.1) has the form

$$y = C_1y_1(t) + C_2y_2(t) \tag{20.3}$$

We begin by considering the initial value problem[1]

$$\left.\begin{aligned} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y'_0 \end{aligned}\right\} \tag{20.4}$$

Suppose that $y_1(t)$ and $y_2(t)$ are both solutions of the homogeneous equation; then

$$y(t) = Ay_1(t) + By_2(t) \tag{20.5}$$

[1]Eq. (20.1) can be put in the same form as (20.4) so long as $a \neq 0$, by setting $p(t) = b/a$ and $q(t) = c/a$. However, (20.4) is considerably more general because we are not requiring the coefficients to be constants.

is also a solution of the homogeneous ODE for all values of A and B. To see if it is possible to find a set of solutions that satisfy the initial condition, (20.4) requires that

$$y(t_0) = Ay_1(t_0) + By_2(t_0) = y_0 \tag{20.6}$$
$$y'(t_0) = Ay_1'(t_0) + By_2'(t_0) = y_0' \tag{20.7}$$

If we multiply the first equation by $y_2'(t_0)$ and the second equation by $y_2(t_0)$ we get

$$Ay_1(t_0)y_2'(t_0) + By_2(t_0)y_2'(t_0) = y_0y_2'(t_0) \tag{20.8}$$
$$Ay_1'(t_0)y_2(t_0) + By_2'(t_0)y_2(t_0) = y_0'y_2(t_0) \tag{20.9}$$

Since the second term in each equation is identical, it disappears when we subtract the second equation from the first:

$$Ay_1(t_0)y_2'(t_0) - Ay_1'(t_0)y_2(t_0) = y_0y_2'(t_0) - y_0'y_2(t_0) \tag{20.10}$$

Solving for A gives

$$A = \frac{y_0y_2'(t_0) - y_0'y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)} \tag{20.11}$$

To get an equation for B we instead multiply (20.6) by $y_1'(t_0)$ and (20.7) by $y_1(t_0)$ to give

$$Ay_1(t_0)y_1'(t_0) + By_2(t_0)y_1'(t_0) = y_0y_1'(t_0) \tag{20.12}$$
$$Ay_1'(t_0)y_1(t_0) + By_2'(t_0)y_1(t_0) = y_0'y_1(t_0) \tag{20.13}$$

Now the coefficients of A are identical, so when we subtract we get

$$By_2(t_0)y_1'(t_0) - By_2'(t_0)y_1(t_0) = y_0y_1'(t_0) - y_0'y_1(t_0) \tag{20.14}$$

Solving for B,

$$B = \frac{y_0y_1'(t_0) - y_0'y_1(t_0)}{y_2(t_0)y_1'(t_0) - y_2'(t_0)y_1(t_0)} \tag{20.15}$$

If we define the **Wronskian** Determinant of any two functions y_1 and y_2 as

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t) \tag{20.16}$$

then

$$A = \frac{1}{W(t_0)}\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}, \qquad B = \frac{1}{W(t_0)}\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix} \tag{20.17}$$

Thus **so long as the Wronskian is non-zero we can solve for A and B.**

Definition 20.1. The **Wronskian Determinant** of two functions y_1 and y_2 is given by

$$W(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t) \tag{20.18}$$

If y_1 and y_2 are a fundamental set of solutions of a differential equation, then $W(t)$ is called the **Wronskian of the Differential Equation.**

Example 20.1. Find the Wronskian of $y_1 = \sin t$ and $y_2 = x^2$.

$$W(y_1, y_2)(t) = y_1 y_2' - y_2 y_1' \tag{20.19}$$
$$= (\sin t)(x^2)' - (x^2)(\sin t)' \tag{20.20}$$
$$= 2x \sin t - x^2 \cos t \quad \Box \tag{20.21}$$

Example 20.2. Find the Wronskian of the differential equation $y'' - y = 0$.

The roots of the characteristic equation is $r^2 - 1 = 0$ are ± 1, and a fundamental pair of solutions are $y_1 = e^t$ and $y_2 = e^{-t}$. The Wronskian is therefore

$$W(x) = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -2. \quad \Box \tag{20.22}$$

The discussion preceding Example (20.1) proved the following theorem.

Theorem 20.2. Existence of Solutions. Let y_1 and y_2 be any two solutions of the equation

$$y'' + p(t)y' + q(t)y = 0 \tag{20.23}$$

such that

$$W(t_0) = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0 \tag{20.24}$$

Then the initial value problem

$$\left.\begin{aligned} y'' + p(t)y' + q(t)y &= 0 \\ y(t_0) &= y_0 \\ y'(t_0) &= y_0' \end{aligned}\right\} \tag{20.25}$$

has a solution, given by (20.17).

Theorem 20.3. General Solution. Suppose that y_1 and y_2 are solutions of

$$y'' + p(t)y' + q(t)y = 0 \tag{20.26}$$

such that for some point t_0 the Wronskian

$$W(t_0) = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0 \tag{20.27}$$

then every solution of (20.26) has the form

$$y(t) = Ay_1(t) + By_2(t) \tag{20.28}$$

for some numbers A and B. In this case y_1 and y_2 are said to form a **fundamental set of solutions** to (20.26).

Theorem 20.4. Let f and g be functions. If their Wronskian is nonzero at some point t_0 then they are linearly independent.

Proof. Suppose that the Wronskian is non-zero at some point t_0. Then

$$W(f,g)(t_0) = \begin{vmatrix} f(t_0) & g(t_0) \\ f'(t_0) & g'(t_0) \end{vmatrix} \neq 0 \tag{20.29}$$

hence

$$f(t_0)g'(t_0) - g(t_0)f'(t_0) \neq 0 \tag{20.30}$$

We will prove the result by contradiction. Suppose that f and g are linearly dependent. Then there exists some non-zero constants A and B such that for all t,

$$Af(t) + Bg(t) = 0 \tag{20.31}$$

Differentiating,

$$Af'(t) + Bg'(t) = 0 \tag{20.32}$$

which holds for all t. Since (20.31) and (20.32) hold for all t, then they hold for $t = t_0$. Hence

$$Af(t_0) + Bg(t_0) = 0 \tag{20.33}$$

$$Af'(t_0) + Bg'(t_0) = 0 \tag{20.34}$$

We can write (20.33) as a matrix:

$$\begin{bmatrix} f(t_0) & g(t_0) \\ f'(t_0) & g'(t_0) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0 \tag{20.35}$$

From linear algebra, since A and B are not both zero, we know that the only way that this can be true is the determinant equals zero. Hence

$$f(t_0)g'(t_0) - g(t_0)f'(t_0) = 0 \tag{20.36}$$

This contradicts equation (20.30), so some assumption we made must be incorrect. The only assumption we made was that f and g were linearly dependent.

Hence f and g must be linearly independent. $\square$

Example 20.3. Show that $y = \sin t$ and $y = \cos t$ are linearly independent.

Their Wronskian is

$$W(t) = (\sin t)(\cos t)' - (\cos t)(\sin t)' \tag{20.37}$$
$$= -\sin^2 t - \cos^2 t \tag{20.38}$$
$$= -1 \tag{20.39}$$

Since $W(t) \neq 0$ for all t then if we pick any particular t, e.g., $t = 0$, we have $W(0) \neq 0$. Hence $\sin t$ and $\cos t$ are linearly independent. □

Example 20.4. Show that $y = \sin t$ and $y = t^2$ are linearly independent.

Their Wronskian is

$$W(t) = (\sin t)(t^2)' - (t^2)(\sin t)' \tag{20.40}$$
$$= 2t \sin t - t^2 \cos t \tag{20.41}$$

At $t = \pi$, we have

$$W(\pi) = 2\pi \sin \pi - \pi^2 \cos \pi = \pi^2 \neq 0 \tag{20.42}$$

Since $W(\pi) \neq 0$, the two functions are linearly independent. □

Corollary 20.5. If f and g are linearly dependent functions, then their Wronskian must be zero at every point t.

Proof. If $W(f, g)(t_0) \neq 0$ at some point t_0 then theorem (20.4) tells us that f and g must be linearly independent. But f and g are linearly dependent, so this cannot happen. Hence their Wronskian can never be nonzero. □

Suppose that y_1 and y_2 are solutions of $y'' + p(t)y' + q(t)y = 0$. Then

$$W(y_1, y_2)(t) = y_1 y_2' - y_2 y_1' \tag{20.43}$$

Differentiating,

$$\frac{d}{dt} W(y_1, y_2)(t) = y_1 y_2'' + y_1' y_2' - y2 y_1'' - y_2' y_1' \tag{20.44}$$
$$= y_1 y_2'' - y_2 y_1'' \tag{20.45}$$

Since y_1 and y_2 are both solutions, they each satisfy the differential equation:

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \tag{20.46}$$
$$y_2'' + p(t)y_2' + q(t)y_2 = 0 \tag{20.47}$$

Multiply the first equation by y_2 and the second equation by y_1,

$$y_1''y_2 + p(t)y_1'y_2 + q(t)y_1y_2 = 0 \tag{20.48}$$
$$y_2''y_1 + p(t)y_2'y_1 + q(t)y_1y_2 = 0 \tag{20.49}$$

Subtracting the first from the second,

$$y_1y_2'' - y_2y_1'' + y_1y_2'p(t) - y_2y_1''p(t) = 0 \tag{20.50}$$

Substituting (20.45),

$$W'(t) = -p(t)W(t) \tag{20.51}$$

This is a separable differential equation in W; the solution is

$$W(t) = C\text{exp}\left(-\int p(t)dt\right) \tag{20.52}$$

This result is know is Abel's Equation or Abel's Formula, and we summarize it in the following theorem.

Theorem 20.6. Abel's Formula. Let y_1 and y_2 be solutions of

$$y'' + p(t)y' + q(t)y = 0 \tag{20.53}$$

where p and q are continuous functions. Then for some constant C,

$$W(y_1, y_2)(t) = C\text{exp}\left(-\int p(t)dt\right) \tag{20.54}$$

Example 20.5. Find the Wronskian of

$$y'' - 2t\sin(t^2)y' + y\sin t = 0 \tag{20.55}$$

up to a constant multiple.

Using Abel's equation,

$$W(t) = C\exp\left(\int 2t\sin(t^2)dt\right) = Ce^{-\cos t^2} \quad \square \tag{20.56}$$

Note that as a consequence of Abel's formula, the only way that W can be zero is if $C = 0$; in this case, it is zero for all t. Thus **the Wronskian of two solutions of an ODE is either always zero or never zero**. If their Wronskian is never zero, by Theorem (20.4), the two solutions must be linearly independent. On the other hand, if the Wronskian is zero at some point t_0 then it is zero at all t, and

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = 0 \tag{20.57}$$

and therefore the system of equations

$$\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0 \tag{20.58}$$

has a solution for A and B where at least one of A and B are non-zero. This means that there exist A and B, at least one of which is non-zero, such that

$$Ay_1(t) + By_2(t) = 0 \tag{20.59}$$

Since this holds for all values of t, y_1 and y_2 are linearly dependent. This proves the following theorem.

Theorem 20.7. Let y_1 and y_2 be solutions of

$$y'' + p(t)y' + q(t)y = 0 \tag{20.60}$$

where p and q are continuous. Then

1. y_1 and y_2 are linearly dependent $\iff$ $W(y_1, y_2)(t) = 0$ for all t.
2. y_1 and y_2 are linearly independent $\iff$ $W(y_1, y_2)(t) \neq 0$ for all t.

We can summarize our results about the Wronskian of solutions and Linear Independence in the following theorem.

Theorem 20.8. Let $y_1(t)$ and $y_2(t)$ be solutions of

$$y'' + p(t)y' + q(t)y = 0 \tag{20.61}$$

Then the following statements are equivalent:

1. y_1 and y_2 form a fundamental set of solutions.
2. y_1 and y_2 are linearly independent.
3. At some point t_0, $W(y_1, y_2)(t_0) \neq 0$.
4. $W(y_1, y_2)(t) \neq 0$ for all t.

Lesson 21

Reduction of Order

The method of reduction of order allows us to find a second solution to the homogeneous equation if we already know one solution. Suppose that y_1 is a solution if

$$y'' + p(t)y' + q(t)y = 0 \tag{21.1}$$

then we look for a solution of the form

$$y_2 = g(t)y_1(t) \tag{21.2}$$

Differentiating twice,

$$y_2' = gy_1' + g'y_1 \tag{21.3}$$

$$y_2'' = gy_1'' + 2g'y_1' + g''y_1 \tag{21.4}$$

Substituting these into (21.1),

$$gy_1'' + 2g'y_1' + g''y_1 + pgy_1' + pg'y_1 + qgy_1 = 0 \tag{21.5}$$

$$2g'y_1' + y_1g'' + pg'y_1 + (y_1'' + py_1' + qy_1)g = 0 \tag{21.6}$$

Since y_1 is a solution of (21.1), the quantity in parenthesis is zero. Hence

$$y_1g'' + (2y_1' + py_1)g' = 0 \tag{21.7}$$

This is a first order equation in $z = g'$,

$$y_1z' + (2y_1' + py_1)z = 0 \tag{21.8}$$

which is separable in z.

$$\frac{1}{z}\frac{dz}{dt} = -\frac{2y_1' + py_1}{y_1} = -2\frac{y_1'}{y_1} - p \tag{21.9}$$

$$\ln z = -2\int \frac{y_1'}{y_1} - \int pdt \tag{21.10}$$

$$= -2\ln y_1 - \int pdt \tag{21.11}$$

$$z = \frac{1}{y_1^2(t)}\exp\left(-\int p(t)dt\right) \tag{21.12}$$

Since $z = g'(t)$,

$$g(t) = \int \left(\frac{1}{y_1^2(t)}\exp\left(-\int p(t)dt\right)\right)dt \tag{21.13}$$

and since $y_2 = gy_1$, the **method of reduction of order** gives

$$\boxed{y_2(t) = y_1(t)\int \left(\frac{1}{y_1^2(t)}\exp\left(-\int p(t)dt\right)\right)dt} \tag{21.14}$$

Example 21.1. Use (21.14) to find a second solution to

$$y'' - 4y' + 4y = 0 \tag{21.15}$$

given that one solution is $y_1 = e^{2t}$.

Of course we already know that since the characteristic equation is $(r-2)^2 = 0$, the root $r = 2$ is repeated and hence a second solution is $y_2 = te^{2t}$. We will now derive this solution with reduction of order.

From equation (21.14), using $p(t) = -4$,

$$y_2(t) = y_1(t)\int \left(\frac{1}{y_1^2(t)}\exp\left(-\int p(t)dt\right)\right)dt \tag{21.16}$$

$$= e^{2t}\int \left(e^{-4t}\exp\left(\int 4dt\right)\right)dt \tag{21.17}$$

$$= e^{2t}\int \left(e^{-4t}e^{4t}\right)dt \tag{21.18}$$

$$= e^{2t}\int dt \tag{21.19}$$

$$= te^{2t} \quad \square \tag{21.20}$$

The method of reduction of order is more generally valid then for equations with constant coefficient which we already know how to solve. It is usually more practical to repeat the derivation rather than using equation (21.14), which is difficult to memorize.

Example 21.2. Find a second solution to

$$y'' + ty' - y = 0 \tag{21.21}$$

using the observation that $y_1 = t$ is a solution.

We look for a solution of the form

$$y_2 = y_1 u = tu \tag{21.22}$$

Differentiating,

$$y_2' = tu' + u \tag{21.23}$$

$$y_2'' = tu'' + 2u' \tag{21.24}$$

Hence from (21.21)

$$tu'' + 2u' + t^2u' + tu - tu = 0 \tag{21.25}$$

$$tu'' + 2u' + t^2u' = 0 \tag{21.26}$$

$$tz' + (2 + t^2)z = 0 \tag{21.27}$$

where $z = u'$. Rearranging and separating variables in z,

$$\frac{1}{z}\frac{dz}{dt} = -\frac{2+t^2}{t} = -\frac{2}{t} - t \tag{21.28}$$

$$\int \frac{1}{z}\frac{dz}{dt}dt = -\int \frac{2}{t}dt - \int tdt \tag{21.29}$$

$$\ln z = -2\ln t - \frac{1}{2}t^2 \tag{21.30}$$

$$z = \frac{1}{t^2}e^{-t^2/2} \tag{21.31}$$

Therefore

$$\frac{du}{dt} = \frac{1}{t^2}e^{-t^2/2} \tag{21.32}$$

or

$$u(t) = \int t^{-2}e^{-t^2/2}dt \tag{21.33}$$

Thus a second solution is

$$y_2 = tu = t\int t^{-2}e^{-t^2/2}dt \tag{21.34}$$

Substitution back into the original ODE verifies that this works. Hence a general solution of the equation is

$$y = At + Bt\int t^{-2}e^{-t^2/2}dt \quad \square \tag{21.35}$$

Example 21.3. Find a second solution of

$$t^2y'' - 4ty' + 6y = 0 \tag{21.36}$$

assuming that $t > 0$, given that $y_1 = t^2$ is already a solution.

We look for a solution of the form $y_2 = uy_1 = ut^2$. Differentiating

$$y_2' = u't^2 + 2tu \tag{21.37}$$

$$y_2'' = u''t^2 + 4tu' + 2u \tag{21.38}$$

Substituting in (21.36),

$$0 = t^2(u''t^2 + 4tu' + 2u) - 4t(u't^2 + 2tu) + 6ut^2 \tag{21.39}$$

$$= t^4u'' + 4t^3u' + 2t^2u - 4t^3u' - 8t^2u + 6t^2u \tag{21.40}$$

$$= t^4u'' \tag{21.41}$$

Since $t > 0$ we can never have $t = 0$; hence we can divide by t^4 to give $u'' = 0$. This means $u = t$. Hence

$$y_2 = uy_1 = t^3 \tag{21.42}$$

is a second solution. $\square$

Example 21.4. Show that

$$y_1 = \frac{\sin t}{\sqrt{t}} \tag{21.43}$$

is a solution of **Bessel's equation of order 1/2**

$$t^2y'' + ty' + \left(t^2 - \frac{1}{4}\right)y = 0 \tag{21.44}$$

and use reduction of order to find a second solution.

Differentiating using the product rule

$$y_1' = \frac{d}{dt}\left(t^{-1/2}\sin t\right) \tag{21.45}$$

$$= t^{-1/2}\cos t - \frac{1}{2}t^{-3/2}\sin t \tag{21.46}$$

$$y_1'' = -t^{-1/2}\sin t - t^{-3/2}\cos t + \frac{3}{4}t^{-5/2}\sin t \tag{21.47}$$

Plugging these into Bessel's equation,

$$t^2\left(-t^{-1/2}\sin t - t^{-3/2}\cos t + \frac{3}{4}t^{-5/2}\sin t\right) + \\ t\left(t^{-1/2}\cos t - \frac{1}{2}t^{-3/2}\sin t\right) + \left(t^2 - \frac{1}{4}\right)t^{-1/2}\sin t = 0 \quad (21.48)$$

$$\left(-t^{3/2}\sin t - t^{-1/2}\cos t + \frac{3}{4}t^{1/2}\sin t\right) + \\ \left(t^{1/2}\cos t - \frac{1}{2}t^{-1/2}\sin t\right) + t^{3/2}\sin t - \frac{1}{4}t^{-1/2}\sin t = 0 \quad (21.49)$$

All of the terms cancel out, verifying that y_1 is a solution.

To find a second solution we look for

$$y_2 = uy_1 = ut^{-1/2}\sin t \quad (21.50)$$

Differentiating

$$y_2' = u't^{-1/2}\sin t - \frac{1}{2}ut^{-3/2}\sin t + ut^{-1/2}\cos t \quad (21.51)$$

$$\begin{aligned} y_2'' &= u''t^{-1/2}\sin t - \frac{1}{2}u't^{-3/2}\sin t + u't^{-1/2}\cos t \\ &\quad - \frac{1}{2}u't^{-3/2}\sin t + \frac{3}{4}ut^{-5/2}\sin t - \frac{1}{2}ut^{-3/2}\cos t \\ &\quad + u't^{-1/2}\cos t - \frac{1}{2}ut^{-3/2}\cos t - ut^{-1/2}\sin t && (21.52) \\ &= u''t^{-1/2}\sin t + u'\left(-t^{-3/2}\sin t + 2t^{-1/2}\cos t\right) + \\ &\qquad u\left(\frac{3}{4}t^{-5/2}\sin t - t^{-3/2}\cos t - t^{-1/2}\sin t\right) && (21.53) \end{aligned}$$

Hence

$$\begin{aligned} 0 &= t^2\Big[u''t^{-1/2}\sin t + u'\left(-t^{-3/2}\sin t + 2t^{-1/2}\cos t\right) + \\ &\quad u\left(\frac{3}{4}t^{-5/2}\sin t - t^{-3/2}\cos t - t^{-1/2}\sin t\right)\Big] \\ &\quad + t\left(u't^{-1/2}\sin t - \frac{1}{2}ut^{-3/2}\sin t + ut^{-1/2}\cos t\right) \\ &\quad + \left(t^2 - \frac{1}{4}\right)ut^{-1/2}\sin t && (21.54) \end{aligned}$$

All of the terms involving u cancel out and we are left with

$$0 = u''t^{3/2}\sin t + 2u't^{3/2}\cos t \tag{21.55}$$

Letting $z = u'$ and dividing through by $t^{3/2}\sin t$,

$$0 = z' + 2z\cot t \tag{21.56}$$

$$\int \frac{dz}{z} = -2\int \cot t dt \tag{21.57}$$

$$\ln z = -2\ln\sin t \tag{21.58}$$

$$z = \sin^{-2} t \tag{21.59}$$

Since $z = u'$ this means

$$\frac{du}{dt} = \frac{1}{\sin^2 t} \implies u = \cot t \tag{21.60}$$

Since $y_2(t) = uy_1(t)$,

$$y_2 = \frac{\cot t\sin t}{\sqrt{t}} = \frac{\cos t}{\sqrt{t}} \tag{21.61}$$

Thus the general solution is

$$y = Ay_1 + By_2 = \frac{A\sin t + B\cos t}{\sqrt{t}} \quad \square \tag{21.62}$$

Method for Reduction of Order. Given a first solution $u(t)$ to $y'' + p(t)y' + q(t)y = 0$, we can find a second linearly independent solution $v(t)$ as follows:

1. Calculate the Wronskian using Abel's formula,

$$W(t) = e^{-\int p(t)dt}$$

2. Calculate the Wronskian directly a second time as

$$W(t) = u'(t)v(t) - v'(t)u(t)$$

3. Set the two expressions equal; the result is a first order differential equation for the second solution $v(t)$.

4. Solve the differential equation for $v(t)$.

5. Then general solution is then $y = Au(t) + Bv(t)$ for some constants A and B.

Example 21.5. Find a second solution to the differential equation

$$ty'' + 10y' = 0 \tag{21.63}$$

given the observation that $y_1(t) = 1$ is a solution.

Since $p(t) = 10/t$, Abel's formula gives

$$W(t) = e^{-\int (10/t)dx} = t^{-10} \tag{21.64}$$

By direct calculation,

$$W(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & y_2 \\ 0 & y_2' \end{vmatrix} = y_2' \tag{21.65}$$

Equating the two expressions for W,

$$y_2' = t^{-10} \tag{21.66}$$

Therefore, $y_2 = -(1/9)t^{-9}$; the general solution to the homogeneous equation is

$$y = C_1y_1 + C_2y_2 = C_1 + C_2t^{-9}. \tag{21.67}$$

Example 21.6. Find a fundamental set of solutions to

$$t^2y'' + 5ty' - 5y = 0 \tag{21.68}$$

given the observation that $y_1 = t$ is one solution.

Calculating the Wronskian directly,

$$W(t) = \begin{vmatrix} t & y_2 \\ 1 & y_2' \end{vmatrix} = ty_2' - y_2 \tag{21.69}$$

From Abel's Formula , since $p(t) = a_1(t)/a_2(t) = 5/t$,

$$W(t) = e^{-\int (5/t)dt} = t^{-5} \tag{21.70}$$

Equating the two expressions and putting the result in standard form,

$$y_2' - (1/t)y_2 = t^{-6} \tag{21.71}$$

An integrating factor is

$$\mu(t) = e^{\int (-1/t)dx} = 1/t \tag{21.72}$$

Therefore

$$y_2' = t\int (1/t)t^{-6}dt = t\left[\frac{t^{-6}}{6}\right] = \frac{1}{5}t^{-5} \tag{21.73}$$

The fundamental set of solutions is therefore $\{t,\ t^{-5}\}$.

Lesson 22

Non-homogeneous Equations with Constant Coefficients

Theorem 22.1. Existence and Uniqueness. The second order linear initial value problem

$$\left.\begin{aligned} a(t)y'' + b(t)y' + c(t)y = f(t) \\ y(t_0) = y_0 \\ y(t_0) = y_1 \end{aligned}\right\} \tag{22.1}$$

has a unique solution, except possibly where $a(t) = 0$. In particular, the second order linear initial value with constant coefficients,

$$\left.\begin{aligned} ay'' + by' + cy = f(t) \\ y(t_0) = y_0 \\ y(t_0) = y_1 \end{aligned}\right\} \tag{22.2}$$

has a unique solution.

We will omit the proof of this for now, since it will follow as an immediate consequence of the more general result for systems we will prove in chapter (26).

Theorem 22.2. Every solution of the differential equation

$$ay'' + by' + cy = f(t) \tag{22.3}$$

has the form

$$y(t) = Ay_{H1}(t) + By_{H2}(t) + y_P(t) \tag{22.4}$$

where y_{H1} and y_{H2} are linearly independent solutions of

$$ay'' + by' + cy = 0 \tag{22.5}$$

and $y_P(t)$ is a solution of (22.3) that is linearly independent of y_{H1} and y_{H2}. Equation (22.4) is called the **general solution of the ODE** (22.3). The two linearly independent solutions of the homogeneous equation are called a **fundamental set of solutions**.

General Concept: To solve the general equation with constant coefficients, we need to find two linearly independent solutions to the homogeneous equation as well as a particular solution to the non-homogeneous solutions.

Theorem 22.3. Subtraction Principle. If $y_{P1}(t)$ and $y_{P2}(t)$ are two different particular solutions of

$$Ly = f(t) \tag{22.6}$$

then

$$y_H(t) = y_{P1}(t) - y_{P2}(t) \tag{22.7}$$

is a solution of the homogeneous equation$Ly = 0$.

Proof. Since

$$Ly_{P1} = f(t) \tag{22.8}$$

$$Ly_{P2} = f(t) \tag{22.9}$$

Then

$$L(y_{P1} - y_{P2}) = Ly_{P1} - Ly_{P2} \tag{22.10}$$

$$= f(t) - f(t) \tag{22.11}$$

$$= 0 \tag{22.12}$$

Hence y_H given by (22.7) satisfies $Ly_H = 0$. □

Theorem 22.4. The linear differential operator

$$Ly = aD^2y + bDy + cy \tag{22.13}$$

can be factored as

$$Ly = (aD^2 + bD + c)y = a(D - r_1)(D - r_2)y \tag{22.14}$$

where r_1 and r_2 are the roots of the characteristic polynomial

$$ar^2 + br + c = 0 \tag{22.15}$$

Proof. Using the quadratic equation, the roots of the characteristic polynomial satisfy

$$r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a} \tag{22.16}$$

$$r_1 r_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a} \tag{22.17}$$

hence

$$b = -a(r_1 + r_2) \tag{22.18}$$

$$c = ar_1 r_2 \tag{22.19}$$

Thus

$$\begin{aligned} Ly &= ay'' + by' + cy \\ &= ay'' - a(r_1 + r_2)y' + ar_1 r_2 y \\ &= a[y'' - (r_1 + r_2)y' + r_1 r_2 y] \\ &= a(D - r_1)(y' - r_2 y) \\ &= a(D - r_1)(D - r_2)y \end{aligned} \tag{22.20}$$

Hence $L = a(D - r_1)(D - r_2)$ which is the desired factorization. □

Theorem (22.4) provides us with the following simple algorithm for solving any second order linear differential equation or initial value problem with constant coefficients.

Algorithm for 2nd Order, Linear, Constant Coefficients IVP

To solve the initial value problem

$$\left.\begin{aligned} ay'' + by' + cy &= f(t) \\ y(t_0) &= y_0 \\ y'(t_0) &= y_1 \end{aligned}\right\} \tag{22.21}$$

1. Find the roots of the characteristic polynomial $ar^2 + br + c = 0$.
2. Factor the differential equation as $a(D - r_1)\underbrace{(D - r_2)y}_{z(t)} = f(t)$.
3. Substitute $z = (D - r_2)y = y' - r_2 y$ to get $a(D - r_1)z = f(t)$.
4. Solve the resulting differential equation $z' - r_1 z = \dfrac{1}{a}f(t)$ for $z(t)$.
5. Solve the first order differential equation $y' - r_2 y = z(t)$ where $z(t)$ is the solution you found in step (4).
6. Solve for arbitrary constants using the initial conditions.

Example 22.1. Solve the initial value problem

$$\left.\begin{aligned} y'' - 10y' + 21y &= 3\sin t \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}\right\} \tag{22.22}$$

The characteristic polynomial is

$$r^2 - 10r + 21 = (r - 3)(r - 7) \tag{22.23}$$

Thus the roots are $r = 3, 7$ and since $a = 1$ (the coefficient of y''), the differential equation can be factored as

$$(D - 3)\underbrace{(D - 7)y}_{z} = 3\sin t \tag{22.24}$$

To solve equation (22.24) we make the substitution

$$z = (D - 7)y = y' - 7y \tag{22.25}$$

Then (22.24) becomes

$$(D - 3)z = 3\sin t \tag{22.26}$$

$$z' - 3z = 3\sin t \tag{22.27}$$

This is a first order linear equation.An integrating factor is $\mu = e^{-3t}$. Multiplying both sides of (22.27) by μ,

$$\frac{d}{dt}\left(ze^{-3t}\right) = 3e^{-3t}\sin t \tag{22.28}$$

Integrating,

$$ze^{-3t} = 3\int e^{-3t}\sin t dt = \frac{3}{10}e^{-3t}(-3\sin t - \cos t) + C \tag{22.29}$$

or

$$z = -\frac{3}{10}(3\sin t + \cos t) + Ce^{-3t} \tag{22.30}$$

Substituting back for $z = y' - 7y$ from equation (22.25) gives us

$$y' - 7y = -\frac{3}{10}(3\sin t + \cos t) + Ce^{3t} \tag{22.31}$$

This is also a first order linear ODE, which we know how to solve. An integrating factor is $\mu = e^{-7t}$. Multiplying (22.31) by μ and integrating over t gives

$$\begin{aligned}
(y' - 7y)(e^{-7t}) &= \left[-\frac{3}{10}(3\sin t + \cos t) + Ce^{3t}\right](e^{-7t}) && (22.32)\\
\frac{d}{dt}\left(ye^{-7t}\right) &= -\frac{3}{10}e^{-7t}(3\sin t + \cos t) + Ce^{-4t} && (22.33)\\
ye^{-7t} &= \int\left[-\frac{3}{10}e^{-7t}(3\sin t + \cos t) + Ce^{-4t}\right]dt && (22.34)\\
&= -\frac{3}{10}\int e^{-7t}(3\sin t + \cos t)dt + C\int e^{-4t}dt && (22.35)
\end{aligned}$$

Integrating the last term would put a -4 into the denominator; absorbing

this into C and including a new constant of integration C' gives

$$\begin{aligned} ye^{-7t} &= -\frac{9}{10}\int e^{-7t}\sin t dt - \frac{3}{10}\int e^{-7t}\cos t t dt + Ce^{-4t} + C' && (22.36)\\ &= -\frac{9}{10}\left(-\frac{1}{50}e^{-7t}(\cos t + 7\sin t)\right) \\ &\qquad -\frac{3}{10}\left(\frac{1}{50}(\sin t - 7\cos t)\right) + Ce^{-4t} + C' && (22.37)\\ &= e^{-7t}\left(\frac{9}{500}\cos t + \frac{63}{500}\sin t - \frac{3}{500}\sin t + \frac{21}{500}\cos t\right) \\ &\qquad + Ce^{-4t} + C' && (22.38)\\ &= \frac{1}{50}e^{-7t}\left(3\cos t + 6\sin t\right) + Ce^{-4t} + C' && (22.39)\\ y &= \frac{1}{50}(3\cos t + 6\sin t) + Ce^{3t} + C'e^{-7t} && (22.40) \end{aligned}$$

The first initial condition, $y(0) = 1$, gives us

$$1 = \frac{3}{50} + C + C' \tag{22.41}$$

To apply the second initial condition, $y'(0) = 0$, we need to differentiate (22.40)

$$y' = \frac{1}{50}(-3\sin t + 6\cos t) + 3Ce^{3t} - 7C'e^{-7t} \tag{22.42}$$

Hence

$$0 = \frac{6}{50} + 3C - 7C' \tag{22.43}$$

Multiplying equation (22.41) by 7

$$7 = \frac{21}{50} + 7C + 7C' \tag{22.44}$$

Adding equations (22.43) and (22.44)

$$7 = \frac{27}{50} + 10C \implies C = \frac{323}{500} \tag{22.45}$$

Substituting this back into any of (22.41), (22.43), or (22.44) gives

$$C' = \frac{147}{500} \tag{22.46}$$

hence the solution of the initial value problem is

$$y = \frac{1}{50}(3\cos t + 6\sin t) + \frac{323}{500}e^{3t} + \frac{147}{500}e^{-7t} \quad \square \tag{22.47}$$

Theorem 22.5. Properties of the Linear Differential Operator. Let

$$L = aD^2 + bD + c \tag{22.48}$$

and denote its characteristic polynomial by

$$P(r) = ar^2 + br + c \tag{22.49}$$

Then for any function y(t) and any scalar r,

$$Ly = P(D)y \tag{22.50}$$

$$Le^{rt} = P(r)e^{rt} \tag{22.51}$$

$$Lye^{rt} = e^{rt}P(D + r)y \tag{22.52}$$

Proof. To demonstrate (22.50) we replace r by D in (22.49):

$$P(D)y = (aD^2 + bD + c)y = Ly \tag{22.53}$$

To derive (22.51), we calculate

$$\begin{aligned}
Le^{rt} &= a(e^{rt})'' + b(e^{rt})' + c(e^{rt} && (22.54)\\
&= ar^2e^{rt} + bre^{rt} + ce^{rt} && (22.55)\\
&= (ar^2 + br + c)e^{rt} && (22.56)\\
&= P(r)e^{rt} && (22.57)
\end{aligned}$$

To derive (22.52) we apply the differential operator to the product ye^{rt} and expand all of the derivatives:

$$\begin{aligned}
Lye^{rt} &= aD^2(ye^{rt}) + bD(ye^{rt}) + cye^{rt} && (22.58)\\
&= a(ye^{rt})'' + b(ye^{rt})' + cye^{rt} && (22.59)\\
&= a(y'e^{rt} + rye^{rt})' + b(y'e^{rt} + rye^{rt}) + cye^{rt} && (22.60)\\
&= a(y''e^{rt} + 2ry'e^{rt} + r^2ye^{rt})\\
&\qquad + b(y'e^{rt} + rye^{rt}) + cye^{rt} && (22.61)\\
&= e^{rt}[a(y'' + 2ry' + r^2y) + b(y' + ry) + cy] && (22.62)\\
&= e^{rt}\left[a(D^2 + 2Dr + r^2)y + b(D + r)y + cy\right] && (22.63)\\
&= e^{rt}\left[a(D + r)^2y + b(D + r)y + cy\right] && (22.64)\\
&= e^{rt}\left[a(D + r)^2 + b(D + r) + c\right] y && (22.65)\\
&= e^{rt}P(D + r)y && (22.66)
\end{aligned}$$

□

Derivation of General Solution. We are now ready to find a general formula for the solution of

$$ay'' + by' + cy = f(t) \tag{22.67}$$

where $a \neq 0$, for any function $f(t)$. We begin by dividing by a,

$$y'' + By' + Cy = q(t) \tag{22.68}$$

where $B = b/a$, $C = c/a$, and $q(t) = f(t)/a$. By the factorization theorem (theorem (22.4)), (22.68) is equivalent to

$$(D - r_1)(D - r_2)y = q(t) \tag{22.69}$$

where

$$r_{1,2} = \frac{1}{2}\left(-B \pm \sqrt{B^2 - 4C}\right) \tag{22.70}$$

Defining

$$z = (D - r_2)y = y' - r_2 y \tag{22.71}$$

equation (22.69) becomes

$$(D - r_1)z = q(t) \tag{22.72}$$

or equivalently,

$$z' - r_1 z = q(t) \tag{22.73}$$

This is a first order linear ODE in $z(t)$ with integrating factor

$$\mu(t) = \exp\left(\int -r_1 dt\right) = e^{-r_1 t} \tag{22.74}$$

and the solution of (22.73) is

$$z = \frac{1}{\mu(t)}\left(\int q(t)\mu(t)dt + C_1\right) \tag{22.75}$$

Using (22.71) this becomes

$$y - r_2 y = p(t) \tag{22.76}$$

where

$$p(t) = \frac{1}{\mu(t)}\left(\int q(t)\mu(t)dt + C_1\right) \tag{22.77}$$

Equation (22.76) is a first order linear ODE in $y(t)$; its solution is

$$y(t) = \frac{1}{\nu(t)}\left(\int p(t)\nu(t)dt + C_2\right) \tag{22.78}$$

where

$$\nu(t) = \exp\left(\int -r_2 dt\right) = e^{-r_2 t} \tag{22.79}$$

Using (22.77) in (22.78),

$$\begin{aligned}
y(t) &= \frac{1}{\nu(t)}\int p(t)\nu(t)dt + \frac{C_2}{\nu(t)} && (22.80)\\
&= \frac{1}{\nu(t)}\int \left[\frac{1}{\mu(t)}\left(\int q(t)\mu(t)dt + C_1\right)\right]\nu(t)dt + \frac{C_2}{\nu(t)} && (22.81)\\
&= \frac{1}{\nu(t)}\int \left[\frac{\nu(t)}{\mu(t)}\left(\int q(t)\mu(t)dt\right) + \frac{C_1\nu(t)}{\mu(t)}\right]dt + \frac{C_2}{\nu(t)} && (22.82)\\
&= \frac{1}{\nu(t)}\int \frac{\nu(t)}{\mu(t)}\left(\int q(t)\mu(t)dt\right)dt + \frac{1}{\nu(t)}\int \frac{C_1\nu(t)}{\mu(t)}dt + \frac{C_2}{\nu(t)} && (22.83)
\end{aligned}$$

Substituting $\mu = e^{-r_1 t}$ and $\nu = e^{-r_2 t}$,

$$y(t) = e^{r_2 t}\int e^{(r_1 - r_2)t}\left(\int q(t)e^{-r_1 t}dt\right)dt + C_1 e^{r_2 t}\int e^{(r_1-r_2)t}dt + C_2 e^{r_2 t} \tag{22.84}$$

If $r_1 \neq r_2$, then

$$\int e^{(r_1-r_2)t}dt = \frac{1}{r_1 - r_2}e^{(r_1-r_2)t} \tag{22.85}$$

and thus (when $r_1 \neq r_2$),

$$y(t) = e^{r_2 t}\int e^{(r_1-r_2)t}\left(\int q(t)e^{-r_1 t}dt\right)dt + C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{22.86}$$

Note that the C_1 in (22.86) is equivalent to the C_1 in (22.88) divided by $(r_1 - r_2)$; since C_1 is arbitrary constant, the $r_1 - r_2$ has been absorbed into it.

If $r_1 = r_2 = r$ in (22.88) (this occurs when $B^2 = 4C$ and hence $r = -B/2 = -b/2a$), then

$$\int e^{(r_1-r_2)t}dt = t \tag{22.87}$$

hence when $r_1 = r_2 = r$,

$$y(t) = e^{rt}\int \left(\int q(t)e^{-rt}dt\right)dt + (C_1 t + C_2)e^{rt} \tag{22.88}$$

Thus the homogeneous solution is

$$y_H = \begin{cases} C_1 e^{r_1 t} + C_2 e^{r_2 t} & \text{if } r_1 \neq r_2 \\ (C_1 t + C_2)e^{rt} & \text{if } r_1 = r_2 = r \end{cases} \tag{22.89}$$

and the particular solution, in either case, is

$$y_P = e^{r_2 t} \int e^{(r_1 - r_2)t} \left(\int q(t) e^{-r_1 t} dt \right) dt \tag{22.90}$$

Returning to the original ODE (22.67), before we defined $q(t) = f(t)/a$,

$$y_P = \frac{1}{a} e^{r_2 t} \int e^{(r_1 - r_2)t} \left(\int f(t) e^{-r_1 t} dt \right) dt \tag{22.91}$$

We have just proven the following two theorems.

Theorem 22.6. The general solution of the second order homogeneous linear differential equation with constant coefficients,

$$ay'' + by' + cy = 0 \tag{22.92}$$

is

$$y_H = \begin{cases} C_1 e^{r_1 t} + C_2 e^{r_2 t} & \text{if } r_1 \neq r_2 \\ (C_1 t + C_2) e^{rt} & \text{if } r_1 = r_2 = r \end{cases} \tag{22.93}$$

where r_1 and r_2 are the roots of the characteristic equation

$$ar^2 + br + c = 0 \tag{22.94}$$

Theorem 22.7. A particular solution for the second order linear differential equation with constant coefficients

$$ay'' + by' + cy = f(t) \tag{22.95}$$

is

$$y_P = \frac{1}{a} e^{r_2 t} \int e^{(r_1 - r_2)t} \left(\int f(t) e^{-r_1 t} dt \right) dt \tag{22.96}$$

where r_1 and r_2 are the roots of the characteristic equation

$$ar^2 + br + c = 0 \tag{22.97}$$

Example 22.2. Solve the initial value problem

$$\left. \begin{aligned} y'' - 9y &= e^{2t} \\ y(0) &= 1 \\ y'(0) &= 2 \end{aligned} \right\} \tag{22.98}$$

The characteristic equation is

$$r^2 - 9 = (r-3)(r+3) = 0 \implies r = \pm 3 \tag{22.99}$$

Thus the solution of the homogeneous equation is

$$y_H = Ae^{3t} + Be^{-3t} \tag{22.100}$$

From equation (22.96) with $a = 1$, $r_1 = 3$, $r_2 = -3$, and $f(t) = e^{2t}$, a particular solution is

$$\begin{aligned}
y_P &= \frac{1}{a}e^{r_2 t}\int e^{(r_1 - r_2)t}\left(\int f(t)e^{-r_1 t}dt\right)dt && (22.101)\\
&= e^{-3t}\int e^{6t}\left(\int e^{2t}e^{-3t}dt\right)dt && (22.102)\\
&= e^{-3t}\int e^{6t}\left(\int e^{-t}dt\right)dt && (22.103)\\
&= -e^{-3t}\int e^{6t}e^{-t}dt && (22.104)\\
&= -e^{-3t}\int e^{5t}dt && (22.105)\\
&= -\frac{1}{5}e^{-3t}e^{5t} && (22.106)\\
&= -\frac{1}{5}e^{2t} && (22.107)
\end{aligned}$$

Hence the general solution is

$$y = Ae^{3t} + Be^{-3t} - \frac{1}{5}e^{2t} \tag{22.108}$$

The first initial condition gives

$$1 = A + B - \frac{1}{5} \implies B = \frac{6}{5} - A \tag{22.109}$$

To apply the second initial condition we must differentiate (22.108):

$$y' = 3Ae^{3t} - 3Be^{-3t} - \frac{2}{5}e^{2t} \tag{22.110}$$

hence

$$2 = 3A - 3B - \frac{2}{5} \implies \frac{12}{5} = 3A - 3B \tag{22.111}$$

From (22.109)

$$\frac{12}{5} = 3A - 3\left(\frac{6}{5} - A\right) \tag{22.112}$$

$$= 6A - \frac{18}{5} \tag{22.113}$$

$$\frac{30}{5} = 6A \tag{22.114}$$

$$A = 1 \tag{22.115}$$

$$B = \frac{6}{5} - A = \frac{1}{5} \tag{22.116}$$

Consequently

$$y = e^{3t} + \frac{1}{5}e^{-3t} - \frac{1}{5}e^{2t} \tag{22.117}$$

is the solution of the initial value problem. $\square$

Lesson 23

Method of Annihilators

In this chapter we will return to using the D operator to represent the derivative operator. In particular, we will be interested the general n^{th} order linear equation with constant coefficients

$$a_n D^n y + a_{n-1} D^{n-1} y + \cdots + a_1 Dy + a_0 y = g(t) \tag{23.1}$$

which we will represent as

$$Ly = g(t) \tag{23.2}$$

where L is the operator

$$L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0 \tag{23.3}$$

As we have seen earlier (see chapter 15) the L operator has the useful property that it can be factored

$$L = (D - r_1)(D - r_2) \cdots (D - r_n) \tag{23.4}$$

where $r_1, \dots, r_n$ are the roots of the characteristic equation

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0 \tag{23.5}$$

Definition 23.1. An operator L is said to be an **annihilator** of a function $f(t)$ if $Lf = 0$.

A solution of a linear homogeneous equation is then any function that can be annihilated by the corresponding differential operator.

Theorem 23.2. D^n annihilates t^{n-1}.

Since

$$D1 = 0 \implies D \text{ annihilates } 1 \tag{23.6}$$
$$D^2 t = 0 \implies D^2 \text{ annihilates } t \tag{23.7}$$
$$D^3 t^2 = 0 \implies D^3 \text{ annihilates } t^2 \tag{23.8}$$
$$\vdots \tag{23.9}$$
$$D^n t^{n-1} = 0 \implies D^n \text{ annihilates } t^{k-1} \tag{23.10}$$

Theorem 23.3. $(D-a)^n$ annihilates $t^{n-1}e^{at}$.

Proof. (induction). For $n = 1$, the theorem states that $D - a$ annihilates e^{at}. To verify this observe that

$$(D-a)e^{at} = De^{at} - ae^{at} = ae^{at} - ae^{at} = 0 \tag{23.11}$$

hence the conjecture is true for $n = 1$.

Inductive step: Assume that $(D-a)^n$ annihilates $t^{n-1}e^{at}$. Thus

$$(D-a)^n t^{n-1} e^{at} = 0 \tag{23.12}$$

Consider

$$(D-a)^{n+1} t^n e^{at} = (D-a)^n (D-a) t^n e^{at} \tag{23.13}$$
$$= (D-a)^n (Dt^n e^{at} - at^n e^{at}) \tag{23.14}$$
$$= (D-a)^n (nt^{n-1} e^{at} + t^n ae^{at} - at^n e^{at}) \tag{23.15}$$
$$= (D-a)^n nt^{n-1} e^{at} \tag{23.16}$$
$$= n(D-a)^n t^{n-1} e^{at} \tag{23.17}$$
$$= 0 \tag{23.18}$$

where the last line follows from (23.12). □

Theorem 23.4. $(D^2 + a^2)$ annihilates any linear combination of $\cos ax$ and $\sin ax$

Proof.

$$(D^2 + a^2)(A \sin at + B \cos at) = AD^2 \sin at + Aa^2 \sin at \tag{23.19}$$
$$+ BD^2 \cos at + a^2 B \cos at \tag{23.20}$$
$$= -Aa^2 \sin at + Aa^2 \sin at + \tag{23.21}$$
$$- Ba^s \cos at + a^2 B \cos at \tag{23.22}$$
$$= 0 \tag{23.23}$$

□

Theorem 23.5. $(D^2 - 2aD + (a^2 + b^2))^n$ annihilates $t^{n-1}e^{at}\cos bt$ and $t^{n-1}e^{at}\sin bt$.

Proof. For n=1.

$$\begin{aligned}
&(D^2 - 2aD + (a^2 + b^2))^1 t^{1-1} e^{at}\sin bt \\
&\quad = (D^2 - 2aD + a^2 + b^2)(e^{at}\sin bt) && (23.24)\\
&\quad = ((D-a)^2 + b^2)(e^{at}\sin bt) && (23.25)\\
&\quad = (D-a)(D-a)(e^{at}\sin bt) + b^2 e^{at}\sin bt && (23.26)\\
&\quad = (D-a)(ae^{at}\sin bt + be^{at}\cos bt - ae^{at}\sin bt) + b^2 e^{at}\sin bt && (23.27)\\
&\quad = (D-a)(be^{at}\cos bt) + b^2 e^{at}\sin bt && (23.28)\\
&\quad = abe^{at}\cos bt - b^2 e^{at}\sin bt - abe^{at}\cos bt + b^2 e^{at}\sin bt && (23.29)\\
&\quad = 0 && (23.30)
\end{aligned}$$

For general n, assume that $(D^2 - 2aD + (a^2 + b^2))^n t^{n-1} e^{at}\cos bt = 0$ and similarly for $\sin bt$. Consider first

$$\begin{aligned}
&(D^2 - 2aD + (a^2 + b^2)) t^n e^{at}\cos bt && (23.31)\\
&\quad = [(D-a)^2 + b^2](t^n e^{at}\cos bt) && (23.32)\\
&\quad = (D-a)^2(t^n e^{at}\cos bt) + b^2(t^n e^{at}\cos bt) && (23.33)\\
&\quad = (D-a)[nt^{n-1}e^{at}\cos bt + at^n e^{at}\cos bt - bt^n e^{at}\sin bt] \\
&\qquad + b^2(t^n e^{at}\cos bt) && (23.34)\\
&\quad = n(n-1)t^{n-2}e^{at}\cos bt + nat^{n-1}e^{at}\cos bt - nbt^{n-1}e^{at}\sin bt \\
&\qquad + nat^{n-1}e^{at}\cos bt + a^2 t^n e^{at}\cos bt - abt^n e^{at}\sin bt \\
&\qquad - bnt^{n-1}e^{at}\sin bt - abt^n e^{at}\sin bt - b^2 t^n e^{at}\cos bt \\
&\qquad - ant^{n-1}e^{at}\cos bt - a^2 t^n e^{at}\cos bt + abt^n e^{at}\sin bt \\
&\qquad + b^2 t^n e^{at}\cos bt && (23.35)\\
&\quad = n(n-1)t^{n-2}e^{at}\cos bt + nat^{n-1}e^{at}\cos bt - nbt^{n-1}e^{at}\sin bt \\
&\qquad - abt^n e^{at}\sin bt - bnt^{n-1}e^{at}\sin bt && (23.36)
\end{aligned}$$

The last line only contains terms such as

$$t^{n-1}e^{at}\cos bt \tag{23.37}$$
$$t^{n-1}e^{at}\sin bt \tag{23.38}$$
$$t^{n-1}e^{at}\cos bt \tag{23.39}$$

But by the inductive hypothesis these are all annihilated by $(D^2 - 2aD + (a^2 + b^2))^n$. Hence $(D^2 - 2aD + (a^2 + b^2))^{n+1}$ annihilates $t^n e^{at} \cos bt$. A similar argument applies to the $\sin bt$ functions, completing the proof by induction. □

Example 23.1. To solve the differential equation

$$y'' - 6y' + 8y = t \tag{23.40}$$

We first solve the homogeneous equation. Its characteristic equation is

$$r^2 - 6y + 8 = 0 \tag{23.41}$$

which has roots at 2 and 4, so

$$y_H = C_1 e^2 t + C_2 e^4 t \tag{23.42}$$

Then we observe that D^2 is an annihilator of t. We rewrite the differential equation as

$$(D^2 - 6D + 8)y = t \tag{23.43}$$

$$D^2(D^2 - 6D - 8)y = D^2 = 0 \tag{23.44}$$

$$\tag{23.45}$$

The characteristic equation is

$$r^2(r-4)(r-2) = 0 \tag{23.46}$$

so the roots are 4,2,0, and 0, giving us additional particular solutions of

$$y_P = At + B \tag{23.47}$$

The general solution is

$$y = C_1 e^2 t + C_2 e^4 t + At + B \tag{23.48}$$

To find A and B we differentiate,

$$y'_P = A \tag{23.49}$$

$$y''_P = 0 \tag{23.50}$$

Substituting into the original differential equation,

$$0 - 6A + 8(At + B) = t \tag{23.51}$$

Equating coefficients gives $A = 1/8$ and $8B = 6A = 3/4 \implies B = 3/32$. Hence

$$y = C_1 e^2 t + C_2 e^4 t + \frac{1}{8}t + \frac{3}{32} \tag{23.52}$$

The constants C_1 and C_2 depend on initial conditions. □

The method of annihilators is really just a variation on the method of undetermined coefficients - it gives you a way to remember or get to the functions you need to remember to get a particular solution. To use it to solve $Ly = g(t)$ you would in general use the following method:

1. Solve the homogeneous equation $Ly = 0$. Call this solution y_H.

2. Operate on both sides of $Ly = g$ with some operator L' so that $L'Ly = L'g = 0$, i.e., using an operator that annihilates g.

3. Find the characteristic equation of $L'Ly = 0$.

4. Solve the homogeneous equation $L'Ly = 0$.

5. Remove the terms in the solution of $L'Ly = 0$ that are linearly dependent on terms in you original solution to $Ly = 0$. The terms that remain are your y_p.

6. Use undetermined coefficients to determine any unknowns in your particular solution.

7. The general solution is $y = y_H + y_p$ where $Ly_H = 0$.

Lesson 24

Variation of Parameters

The **method of variation of parameters** gives an explicit formula for a particular solution to a linear differential equation once all of the homogeneous solutions are known. The particular solution is a **pseudo-linear combination** of the homogeneous equation. By a pseudo-linear combination we mean an expression that has the same form as a linear combination, but the constants are allowed to depend on t:

$$y_P = u_1(t)y_1 + u_2(t)y_2 \tag{24.1}$$

where $u_1(t)$ and $u_(t)$ are unknown functions of t that are treated as parameters. The name of the method comes from the fact that the parameters (the functions u_1 and u_2 in the linear combination) are allowed to vary.

For the method of variation of parameters to work **we must already know two linearly independent solutions to the homogeneous equation**s Suppose that $y_1(t)$ and $y_2(t)$ are linearly independent solutions of

$$a(t)y'' + b(t)y' + c(t)y = 0 \tag{24.2}$$

The idea is to look for a pair of functions $u(t)$ and $v(t)$ that will make

$$y_P = u(t)y_1 + v(t)y_2 \tag{24.3}$$

a solution of

$$a(t)y'' + b(t)y' + c(t)y = f(t). \tag{24.4}$$

Differentiating equation (24.3)

$$y'_P = u'y_1 + uy'_1 + v'y_2 + vy'_2 \tag{24.5}$$

If we now make the totally arbitrary assumption that

$$u'y_1 + v'y_2 = 0 \tag{24.6}$$

then

$$y_P' = uy_1' + vy_2' \tag{24.7}$$

and therefore

$$y_P'' = u'y_1' + uy_1'' + v'y_2' + vy_2'' \tag{24.8}$$

From equation (24.4)

$$\begin{aligned} f(t) &= a(t)(u'y_1' + uy_1'' + v'y_2' + vy_2'') \\ &\quad + b(t)\,(uy_1' + vy_2') + c(t)\,(uy_1 + vy_2) && (24.9) \\ &= a(t)\,(u'y_1' + v'y_2') + u\,[a(t)y_1'' + b(t)y_1' + c(t)y_1] \\ &\quad + v[a(t)y_2'' + b(t)y_2' + c(t)y_2] && (24.10) \\ &= a(t)(u'y_1' + v'y_2') && (24.11) \end{aligned}$$

Combining equations (24.6) and (24.11) in matrix form

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f(t)/a(t) \end{pmatrix} \tag{24.12}$$

The matrix on the left hand side of equation (24.12) is the Wronskian, which we know is nonsingular, and hence invertible, because y_1 and y_2 form a fundamental set of solutions to a differential equation. Hence

$$\begin{aligned} \begin{pmatrix} u' \\ v' \end{pmatrix} &= \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ f(t)/a(t) \end{pmatrix} \\ &= \frac{1}{W(t)} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ f(t)/a(t) \end{pmatrix} && (24.13) \\ &= \frac{1}{W(t)} \begin{pmatrix} -y_2 f(t)/a(t) \\ y_1 f(t)/a(t) \end{pmatrix} \end{aligned}$$

where $W(t) = y_1y_2' - y_2y_1'$. Hence

$$\frac{du}{dt} = \frac{-y_2 f(t)}{a(t)W(t)} \tag{24.14}$$

$$\frac{dv}{dt} = \frac{y_1 f(t)}{a(t)W(t)} \tag{24.15}$$

Integrating each of these equations,

$$u(t) = -\int \frac{y_2(t)f(t)}{a(t)W(t)}dt \tag{24.16}$$

$$v(t) = \int \frac{y_1(t)f(t)}{a(t)W(t)}dt \tag{24.17}$$

Substituting into equation (24.3)

$$\boxed{y_P = -y_1 \int \frac{y_2 f(t)}{a(t)W(t)}dt + y_2 \int \frac{y_1 f(t)}{a(t)W(t)}dt} \tag{24.18}$$

which is the basic equation of the method of variation of parameters. It is usually easier to reproduce the derivation, however, then it is to remember the solution. This is illustrated in the following examples.

Example 24.1. Find the general solution to $y'' - 5y' + 6y = e^t$

The characteristic polynomial $r^2 - 5r + 6 = (r-3)(r-2) = 0$, hence a fundamental set of solutions is $y_1 = e^{3t}$, $y_2 = e^{2t}$. The Wronskian is

$$W(t) = \begin{vmatrix} e^{3t} & e^{2t} \\ 3e^{3t} & 2e^{2t} \end{vmatrix} = -e^{5t} \tag{24.19}$$

Since $f(t) = e^t$ and $a(t) = 1$,

$$y_P = -e^{3t} \int \frac{e^{2t}e^t}{-e^{5t}}dt + e^{2t} \int \frac{e^{3t}e^t}{-e^{5t}}dt \tag{24.20}$$

$$= e^{3t} \int e^{-2t}dt - e^{2t} \int e^{-t}dt \tag{24.21}$$

$$= -\frac{1}{2}e^t + e^t = \frac{1}{2}e^t \tag{24.22}$$

(We ignore any constants of integration in this method). Thus the general solution is

$$y = y_P + y_H = \frac{1}{2}e^t + C_1 e^{3t} + C_2 e^{2t} \quad \square \tag{24.23}$$

Example 24.2. Find a particular solution to $y'' - 5y' + 6y = t$ by repeating the steps in the derivation of (24.18) rather than plugging in the general formula.

From the previous example we have homogeneous solutions $y_1 = e^{3t}$ and $y_2 = e^{2t}$. Therefore we look for a solution of the form

$$y = u(t)e^{3t} + v(t)e^{2t} \tag{24.24}$$

Differentiating,

$$y' = u'(t)e^{3t} + 3u(t)e^{3t} + v'(t)e^{2t} + 2v(t)e^{2t} \tag{24.25}$$

Assuming that the sum of the terms with the derivatives of u and v is zero,

$$u'(t)e^{3t} + v'(t)e^{2t} = 0 \tag{24.26}$$

and consequently

$$y' = 3u(t)e^{3t} + 2v(t)e^{2t} \tag{24.27}$$

Differentiating,

$$y'' = 3u'(t)e^{3t} + 9u(t)e^{3t} + 2v'(t)e^{2t} + 4v(t)e^{2t} \tag{24.28}$$

Substituting into the differential equation $y'' - 5y' + 6y = t$,

$$\begin{aligned} t =& (3u'(t)e^{3t} + 9u(t)e^{3t} + 2v'(t)e^{2t} + 4v(t)e^{2t}) - 5(3u(t)e^{3t} + 2v(t)e^{2t}) \\ &+ 6(u(t)e^{3t} + v(t)e^{2t}) && (24.29) \\ =& 3u(t)'e^{3t} + 2v(t)'e^{2t} && (24.30) \end{aligned}$$

Combining our results gives (24.26) and (24.30) gives

$$u'(t)e^{3t} + v'(t)e^{2t} = 0 \tag{24.31}$$

$$3u(t)'e^{3t} + 2v'(t)e^{2t} = t \tag{24.32}$$

Multiplying equation (24.31) by 3 and subtracting equation (24.32) from the result,

$$v'(t)e^{2t} = -t \tag{24.33}$$

We can solve this by multiplying through by e^{-2t} and integrating,

$$v(t) = -\int te^{-2t}dt = -\left(-\frac{t}{2} - \frac{1}{4}\right)e^{-2t} = \left(\frac{t}{2} + \frac{1}{4}\right)e^{-2t} \tag{24.34}$$

because $\int te^{at}dt = \left[t/a - 1/a^2\right]e^{at}$.

Multiplying equation (24.31) by 2 and subtracting equation (24.32) from the result,

$$u'(t)e^{3t} = t \tag{24.35}$$

which we can solve by multiplying through by e^{-3t} and integrating:

$$u(t) = \int te^{-3t}dt = \left(-\frac{t}{3} - \frac{1}{9}\right)e^{-3t} = -\left(\frac{t}{3} + \frac{1}{9}\right)e^{-3t} \tag{24.36}$$

Thus from (24.24)

$$\begin{aligned} y &= u(t)e^{3t} + v(t)e^{2t} && (24.37) \\ &= \left(-\left(\frac{t}{3} + \frac{1}{9}\right)e^{-3t}\right)e^{3t} + \left(\left(\frac{t}{2} + \frac{1}{4}\right)e^{-2t}\right)e^{2t} && (24.38) \\ &= -\frac{t}{3} - \frac{1}{9} + \frac{t}{2} + \frac{1}{4} && (24.39) \\ &= \frac{t}{6} + \frac{5}{36} \quad \square && (24.40) \end{aligned}$$

Example 24.3. Solve the initial value problem

$$\left.\begin{aligned} t^2y'' - 2y = 2t \\ y(1) = 0 \\ y'(1) = 1 \end{aligned}\right\} \tag{24.41}$$

given the observation that $y = t^2$ is a homogeneous solution.

First, we find a second homogeneous solution using reduction of order. By Abel's formula, since there is no y' term, the Wronskian is a constant:

$$W(t) = e^{-\int 0 dt} = C \tag{24.42}$$

A direct calculation of the Wronskian gives

$$W(t) = \begin{vmatrix} t^2 & y_2 \\ 2t & y_2' \end{vmatrix} = t^2y_2' - 2ty_2 \tag{24.43}$$

Setting the two expressions for the Wronskian equal to one another gives

$$t^2y' - 2ty = C \tag{24.44}$$

where we have omitted the subscript. This is a first order linear equation in y;putting it into standard form,

$$y' - \frac{2}{t}y = \frac{C}{t^2} \tag{24.45}$$

An integrating factor is $\mu = e^{\int(-2/t)dt} = e^{-2\ln t} = t^{-2}$, hence

$$\frac{d}{dt}\frac{y}{t^2} = Ct^{-4} \tag{24.46}$$

Integrating both sides of the equation over t,

$$\frac{y}{t^2} = -\frac{C}{3}t^{-3} + C_1 \tag{24.47}$$

Multiplying through by t^2,

$$y = C_1t^2 + \frac{C_2}{t} \tag{24.48}$$

where $C_2 = C/-3$. Since $y_1 = t^2$, we conclude that $y_2 = 1/t$. This gives us the entire homogeneous solution.

Next, we need to find a particular solution; we can do this using the variation of parameters formula. Since $a(t) = t^2$ and $f(t) = 2t$, we have

$$y_P = -y_1(t) \int \frac{y_2(t)f(t)}{a(t)W(t)} dt + y_2(t) \int \frac{y_1(t)f(t)}{a(t)W(t)} dt \tag{24.49}$$

We previously had $W = C$; but here we need an exact value. To get the exact value of C, we calculate the Wronskina from the now-known homogeneous solutions,

$$C = W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^2 & 1/t \\ 2t & -1/t^2 \end{vmatrix} = -3 \tag{24.50}$$

Using $y_1 = t^2$, $y_2 = 1/t$, $f(t) = 2t$, and $a(t) = t^2$,

$$\begin{aligned} y_P &= -t^2 \int \frac{2t}{t \cdot t^2 \cdot -3} dt + \frac{1}{t} \int \frac{t^2 \cdot 2t}{t^2 \cdot -3} dt && (24.51) \\ &= \frac{2t^2}{3} \int t^{-2} dt - \frac{2}{3t} \int t dt && (24.52) \\ &= \frac{2t^2}{3} \cdot \frac{-1}{t} - \frac{2}{3t} \cdot \frac{t^2}{2} && (24.53) \\ &= -\frac{2t}{3} - \frac{t}{3} = -t && (24.54) \end{aligned}$$

Hence $y_P = -t$ and therefore

$$y = y_H + y_P = C_1 t^2 + C_2 t - \frac{1}{t} \tag{24.55}$$

From the first initial condition we have $0 = C_1 + C_2 - 1$ or

$$C_1 + C_2 = 1. \tag{24.56}$$

To use the second condition we need the derivative,

$$y' = 2C_1 t - \frac{C_2}{t^2} - 1 \tag{24.57}$$

hence the second condition gives $1 = 2C_1 - C_2 - 1$ or

$$2C_1 - C_2 = 2. \tag{24.58}$$

Solving for C_1 and C_2 gives $C_1 = 1$, hence $C_2 = 0$, so that the complete solution of the initial value problem is

$$y = t^2 - \frac{1}{t} \quad \square \tag{24.59}$$

Lesson 25

Harmonic Oscillations

If a spring is extended from its resting length by an amount y then a restoring force, opposite in direction from but proportional to the displacement will attempt to pull the spring back; the subsequent motion of an object of mass m attached to the end of the spring is described by Newton's laws of motion:

$$my'' = -ky \tag{25.1}$$

where k is a positive constant that is determined by the mechanical properties of the spring. The right-hand side of quation (25.1) – that the force is proportional to the displacement – is known as **Hooke's Law**.

Simple Harmonic Motion

Rearranging equation (25.1),

$$y'' + \omega^2 y = 0 \tag{25.2}$$

where $\omega = \sqrt{k/m}$ is called the **oscillation frequency**. Equation (25.2) is called the **simple harmonic oscillator** equation because there are no additional drag or forcing functions. The oscillator, once started, continues to oscillate forever in this model. There are no physical realizations of (25.2) in nature because there is always some amount of drag. To keep a spring moving we need to add a motor. Before we see how to describe drag and forcing functions we will study the simple oscillator.

The characteristic equation is $r^2 + \omega^2 = 0$, and since the roots are purely imaginary ($r = \pm i\omega$) the motion is oscillatory,

$$y = C_1 \cos \omega t + C_2 \sin \omega t \tag{25.3}$$

It is sometimes easier to work with a single trig function then with two. To do this we start by defining the parameter

$$A^2 = C_1^2 + C_2^2 \tag{25.4}$$

where we chose the positive square root for A, and define the angle ϕ such that

$$\cos \phi = \frac{C_1}{A} \tag{25.5}$$

$$\sin \phi = -\frac{C_2}{A} \tag{25.6}$$

We know such an angle exists because $0 \leq |C_1| \leq A$ and $0 \leq |C_2| \leq A$ and by (25.4) ϕ satisfies the identity $\cos^\phi + \sin^2 \phi = 1$ as required. Thus

$$y = A \cos \phi \cos \omega t - A \sin \phi \sin \omega t \tag{25.7}$$

$$= A \cos(\phi + \omega t) \tag{25.8}$$

Then $\phi = \tan^{-1}(C_1/C_2)$ is known as the **phase** of the oscillations and A is called the **amplitude**. As we see, the oscillation is described by a single sine wave of magnitude (height) A and phase shift ϕ. With a suitable redefinition of C_1 and C_2, we could have made the cos into sin rather than a cosine, e.g., $C_1/A = \sin \phi$ and $C_2/phi = \cos \phi$.

Damped Harmonic Model

In fact, equation (25.2) is not such a good model because it predicts the system will oscillate indefinitely, and not slowly damp out to zero. A good approximation to the damping is a force that acts linearly against the motion: the faster the mass moves, the stronger the damping force. Its direction is negative, since it acts against the velocity y' of the mass. Thus we modify equation (25.2) to the following

$$my'' = -Cy' - ky \tag{25.9}$$

where $C \geq 0$ is a damping constant that takes into account a force that is proportional to the velocity but acts in the opposite direction to the velocity.

It is standard to define a new constant $b = C/m$ and a frequency $\omega = \sqrt{k/m}$ as before so that we can rearrange our equation into standard form as

$$y'' + by' + \omega^2 y = 0 \tag{25.10}$$

Equation (25.10) is the standard equation of **damped harmonic motion**. As before, it is a linear second order equation with constant coefficients, so we can solve it exactly by finding the roots of the characteristic equation

$$r^2 + br + \omega^2 = 0 \tag{25.11}$$

The roots of the characteristic equation are given by the quadratic equation as

$$r = \frac{-b \pm \sqrt{b^2 - 4\omega^2}}{2} \tag{25.12}$$

The resulting system is said to be

- **underdamped** when $b < 2\omega$;
- **critically damped** when $b = 2\omega$; and
- **overdamped** when $b > 2\omega$.

In the underdamped system ($b < 2\omega$) the roots are a complex conjugate pair with negative real part, $r = \mu \pm i\varpi$ where

$$\mu = b/2 > 0, \quad \varpi = \omega\sqrt{1 - (b/2\omega)^2} \tag{25.13}$$

and hence the resulting oscillations are described by decaying oscillations

$$y = Ae^{-\mu t}\sin(\varpi t + \phi) \tag{25.14}$$

The critically damped system has a single positive real repeated root $r = \mu = b/2$, so that

$$y = (C_1 + C_2 t)e^{-\mu t} \tag{25.15}$$

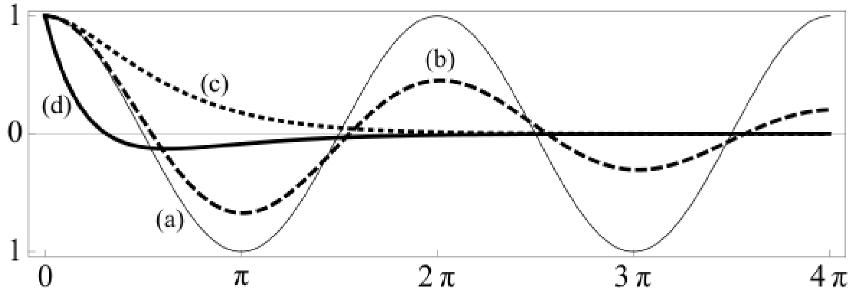
The critically damped systems decays directly to zero without crossing the y-axis.

The overdamped system has two negative real roots $-\mu \pm |\varpi|$ (where $|\varpi| < \mu$) and hence

$$y = e^{-\mu t}\left(C_1 e^{|\varpi| t} + C_2 e^{-|\varpi| t}\right) \tag{25.16}$$

The system damps to zero without oscillations, but may cross the y-axis once. The first term in parenthesis in (25.16) does not give an exponential increase because $|\varpi| < \mu$.

Figure 25.1: Harmonic oscillations with $\omega = 1$ and initial conditions $y(0) = 1$, $y'(0) = 0$. (a) simple harmonic motion; (b) under-damped system with $b = 0.25$; (c) critically damped system with $b = 2.0$; (d) under-damped system with $b = 0.025$.



Forced Oscillations

Finally, it is possible to imaging adding a motor to the spring that produces a force $f(t)$. The resulting is system, including damping, called **the forced harmonic oscillator**:

$$my'' = -Cy'' - ky + f(t) \tag{25.17}$$

If we define a force per unit mass $F(t) = f(t)/m$ and ω and the parameter b as before, this becomes, in standard form,

$$y'' + by' + \omega^2 y = F(t) \tag{25.18}$$

This is the general second-order linear equation with constant coefficients that are positive. Thus any second order linear differential equation with positive constant coefficients describes a harmonic oscillator of some sort. The particular solution is

$$y_P = e^{r_2 t} \int_t e^{(r_1 - r_2)u} \int_u e^{-r_1 s} F(s) ds \tag{25.19}$$

where r_1 and r_2 are the roots of the characteristic equation. For example, suppose the system driven by a force function

$$F(t) = F_0 \sin \alpha t \tag{25.20}$$

Then

$$y_P = F_0 e^{r_2 t} \int_t e^{(r_1 - r_2)u} \int_u e^{-r_1 s} \sin \alpha s \; ds du \tag{25.21}$$

$$= \frac{F_0}{\alpha^2 + r_1^2} e^{r_2 t} \int_t e^{-r_2 u} (\alpha \cos \alpha u + r_1 \sin \alpha u) du \tag{25.22}$$

As with the unforced case, we can define the amplitude and phase angle by

$$A\sin\theta = -\alpha(r_1 + r_2) \tag{25.23}$$

$$A\cos\theta = \alpha^2 - r_1 r_2 \tag{25.24}$$

Then

$$y_P = \frac{F_0 A \sin(\alpha t + \theta)}{(\alpha^2 + r_1^2)(\alpha^2 + r_2^2)} \tag{25.25}$$

where

$$A^2 = [-\alpha(r_1 + r_2)]^2 + [\alpha^2 - r_1 r_2]^2 \tag{25.26}$$

$$= \alpha^2 b^2 + (\alpha^2 - \omega^2)^2 \tag{25.27}$$

because $r_1 + r_2 = -b$ and $r_1 r_2 = \omega^2$. Furthermore,

$$(\alpha^2 + r_1^2)(\alpha^2 + r_2^2) \quad = \quad \alpha^4 + (r_1^2 + r_2^2)\alpha^2 + (r_1 r_2)^2 \tag{25.28}$$

and therefore

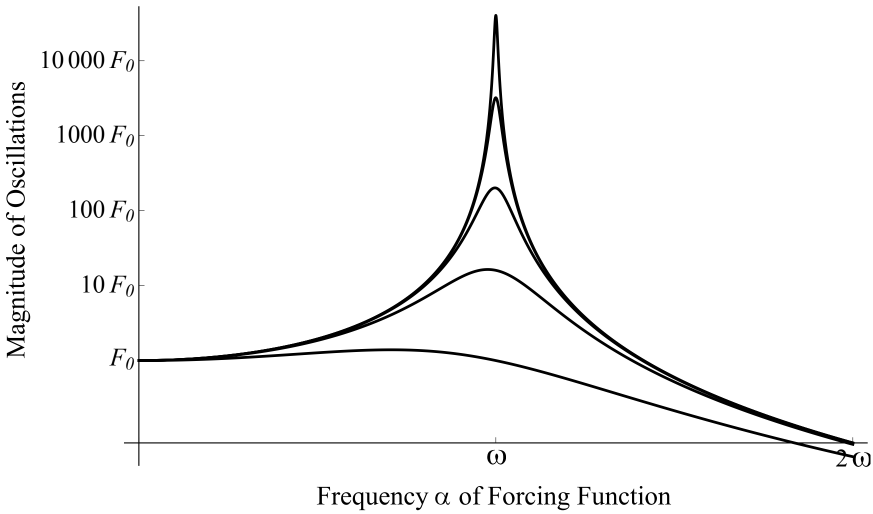
$$y_P = \frac{F_0 \sin(\alpha t + \theta)}{\sqrt{\alpha^2 b^2 + (\alpha^2 - \omega^2)^2}} \tag{25.29}$$

Forcing the oscillator pumps energy into the system; it has a maximum at $\alpha = \omega$, which is unbounded (infinite) in the absence of damping. This phenomenon – that the magnitude of the oscillations is maximized when the system is driven at its natural frequency – is known as resonance. If there is any damping at all the homogeneous solutions decay to zero and all that remains is the particular solution – so the resulting system will eventually be strongly dominated by (25.29), oscillating in synch with the driver. If there is no damping ($b = 0$) then

$$y = C\sin(\omega t + \phi) + \frac{F_0 \sin(\alpha t + \theta)}{|\alpha^2 - \omega^2|} \tag{25.30}$$

where C is the natural magnitude of the system, determined by its initial conditions.

Figure 25.2: The amplitude of oscillations as a function of the frequency of the forcing function, α, as given by (25.29), is shown for various values of the damping coefficient $b = 1,\ 0.3,\ 0.1,\ 0.03,\ 0.01$ (bottom to top) with $\omega = 1$.The oscillations resonate as $\alpha \to \omega$.



Lesson 26

General Existence Theory*

In this section we will show that convergence of Picard Iteration is the equivalent of finding the fixed point of an operator in a general linear vector space. This allows us to expand the scope of the existence theorem to initial value problems involving differential equations of any order as well systems of differential equations. This section is somewhat more theoretical and may be skipped without any loss of continuity in the notes.

Before we look at fixed points of operators we will first review the concept of fixed points of functions.

Definition 26.1. Fixed Point of a Function. Let $f : \mathbb{R} \mapsto \mathbb{R}$. A number $a \in \mathbb{R}$ is called a fixed point of f if $f(a) = a$.

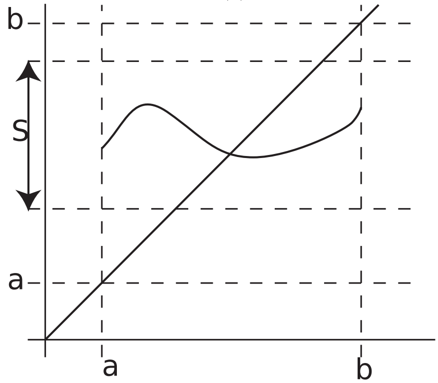
Example 26.1. Find the fixed points of the function $f(x) = x^4 + 2x^2 + x - 3$.

$$\begin{aligned} x &= x^4 + 2x^2 + x - 3 \\ 0 &= x^4 + 2x^2 - 3 \\ &= (x-1)(x+1)(x^2+3) \end{aligned}$$

Hence the real fixed points are $x = 1$ and $x = -1$. $\square$

A function $f : \mathbb{R} \mapsto \mathbb{R}$ has a fixed point if and only if its graph intersects with the line $y = x$. If there are multiple intersections, then there are multiple fixed points. Consequently a sufficient condition is that the range of f is contained in its domain (see figure 26.1).

Figure 26.1: A sufficient condition for a bounded continuous function to have a fixed point is that the range be a subset of the domain. A fixed point occurs whenever the curve of $f(t)$ intersects the line $y = t$.



Theorem 26.2 (Sufficient condition for fixed point). Suppose that $f(t)$ is a continuous function that maps its domain into a subset of itself, i.e.,

$$f(t) : [a, b] \mapsto S \subset [a, b] \tag{26.1}$$

Then $f(t)$ has a fixed point in $[a, b]$.

Proof. If $f(a) = a$ or $f(b) = b$ then there is a fixed point at either a or b. So assume that both $f(a) \neq a$ and $f(b) \neq b$. By assumption, $f(t) : [a, b] \mapsto S \subset [a, b]$, so that

$$f(a) \geq a \quad \text{and} \quad f(b) \leq b \tag{26.2}$$

Since both $f(a) \neq a$ and $f(b) \neq b$, this means that

$$f(a) > a \quad \text{and} \quad f(b) < b \tag{26.3}$$

Let $g(t) = f(t) - t$. Then g is continuous because f is continuous, and furthermore,

$$g(a) = f(a) - a > 0 \tag{26.4}$$

$$g(b) = f(b) - b < 0 \tag{26.5}$$

Hence by the intermediate value theorem, g has a root $r \in (a, b)$, where $g(r) = 0$. Then

$$0 = g(r) = f(r) - r \implies f(r) = r \tag{26.6}$$

i.e., r is a fixed point of f. □

In the case just proven, there may be multiple fixed points. If the derivative is sufficiently bounded then there will be a unique fixed point.

Theorem 26.3 (Condition for a unique fixed point). Let f be a continuous function on $[a, b]$ such that $f : [a, b] \mapsto S \subset (a, b)$, and suppose further that there exists some postive constant $K < 1$ such that

$$|f'(t)| \leq K, \quad \forall t \in [a, b] \tag{26.7}$$

Then f has a unique fixed point in $[a, b]$.

Proof. By theorem 26.2 a fixed point exists. Call it p,

$$p = f(p) \tag{26.8}$$

Suppose that a second fixed point $q \in [a, b]$, $q \neq p$ also exists, so that

$$q = f(q) \tag{26.9}$$

Hence

$$|f(p) - f(q)| = |p - q| \tag{26.10}$$

By the mean value theorem there is some number c between p and q such that

$$f'(c) = \frac{f(p) - f(q)}{p - q} \tag{26.11}$$

Taking absolute values,

$$\left| \frac{f(p) - f(q)}{p - q} \right| = |f'(c)| \leq K < 1 \tag{26.12}$$

and thence

$$|f(p) - f(q)| < |p - q| \tag{26.13}$$

This contradicts equation 26.10. Hence our assumption that a second, different fixed point exists must be incorrect. Hence the fixed point is unique. □

Theorem 26.4 (Fixed Point Iteration Theorem). Let f be as defined in theorem 26.3, and $p_0 \in (a, b)$. Then the sequence of numbers

$$\left.\begin{array}{c} p_1 = f(p_0) \\ p_2 = f(p_1) \\ \vdots \\ p_n = f(p_{n-1}) \\ \vdots \end{array}\right\} \tag{26.14}$$

converges to the unique fixed point of f in (a, b).

Proof. We know from theorem 26.3 that a unique fixed point p exists. We need to show that $p_i \to p$ as $i \to \infty$.

Since f maps onto a subset of itself, every point $p_i \in [a,b]$.

Further, since p itself is a fixed point, $p = f(p)$ and for each i, since $p_i = f(p_{i-1})$, we have

$$|p_i - p| = |p_i - f(p)| = |f(p_{i-1}) - f(p)| \tag{26.15}$$

If for any value of i we have $p_i = p$ then we have reached the fixed point and the theorem is proved.

So we assume that $p_i \neq p$ for all i.

Then by the mean value theorem, for each value of i there exists a number c_i between p_{i-1} and p such that

$$|f(p_{i-1}) - f(p)| = |f'(c_i)||p_{i-1} - p| \leq K|p_{i-1} - p| \tag{26.16}$$

where the last inequality follows because f' is bounded by $K < 1$ (see equation 26.7).

Substituting equation 26.15 into equation 26.16,

$$|p_i - p| = |f(p_{i-1}) - f(p)| \leq K|p_{i-1} - p| \tag{26.17}$$

Restating the same result with i replaced by $i-1, i-2, \dots,$

$$\left.\begin{array}{r}
|p_{i-1} - p| = |f(p_{i-2}) - f(p)| \leq K|p_{i-2} - p| \\
|p_{i-2} - p| = |f(p_{i-3}) - f(p)| \leq K|p_{i-3} - p| \\
|p_{i-3} - p| = |f(p_{i-4}) - f(p)| \leq K|p_{i-4} - p| \\
\vdots \\
|p_2 - p| = |f(p_1) - f(p)| \leq K|p_1 - p| \\
|p_1 - p| = |f(p_0) - f(p)| \leq K|p_0 - p|
\end{array}\right\} \tag{26.18}$$

Putting all these together,

$$|p_i - p| \leq K^2|p_{i-2} - p| \leq K^3|p_{i-2} - p| \leq \cdots \leq K^i|p_0 - p| \tag{26.19}$$

Since $0 < K < 1$,

$$0 \leq \lim_{i\to\infty} |p_i - p| \leq |p_0 - p| \lim_{i\to\infty} K^i = 0 \tag{26.20}$$

Thus $p_i \to p$ as $i \to \infty$. □

Theorem 26.5. Under the same conditions as theorem 26.4 except that the condition of equation 26.7 is replaced with the following condition: $f(t)$ is Lipshitz with Lipshitz constant $K < 1$. Then fixed point iteration converges.

Proof. Lipshitz gives equation 26.16. The rest of the the proof follows as before. □

The Lipshitz condition can be generalized to apply to functions on a vector space.

Definition 26.6. Lipshitz Condition on a Vector Space. Let $\mathcal{V}$ be a vector space and let $t \in \mathbb{R}$. Then $f(t, y)$ is Lipshitz if there exists a real constant K such that

$$|f(t, y) - f(t, z)| \leq K|y, z| \tag{26.21}$$

for all vectors $y, z \in \mathcal{V}$.

Definition 26.7. Let $\mathcal{V}$ be a normed vector space, $S \subset \mathcal{V}$. A **contraction** is any mapping $T : S \mapsto \mathcal{V}$ such that

$$\|Ty - Tz\| \leq K\|y - z\| \tag{26.22}$$

where $0 < K < 1$, holds for all $y, z \in S$. We will call the number K the **contraction constant**. Observe that *a contraction is analogous to a Lipshitz condition on operators with* $K < 1$.

We will need the following two results from analysis:

1. **A Cauchy Sequence** is a sequence $y_0, y_1, \dots$ of vectors in $\mathcal{V}$ such that $\|v_m - v_n\| \to 0$ as $n, m \to \infty$.

2. **Complete Vector Field**. If every Cauchy Sequence converges to an element of $\mathcal{V}$, then we call $\mathcal{V}$ complete.

The following lemma plays the same role for contractions that Lemma (12.6) did for functions.

Lemma 26.8. Let T be a contraction on a complete normed vector space $\mathcal{V}$ with contraction constant K. Then for any $y \in \mathcal{V}$

$$\|T^n y - y\| \leq \frac{1 - K^n}{1 - K}\|Ty - y\| \tag{26.23}$$

Proof. Use induction. For $n = 1$, the formula gives

$$\|Ty - y\| \leq \frac{1-K}{1-K}\|Ty - y\| = \|Ty - y\| \tag{26.24}$$

which is true.

For $n > 1$ suppose that equation 26.23 holds. Then

$$\begin{aligned}
\|T^{n+1}y - y\| &= \|T^{n+1}y - T^n y + T^n y - y\| && (26.25)\\
&\leq \|T^{n+1}y - T^n y\| + \|T^n y - y\| \quad \text{(triangle ineqality)} && (26.26)\\
&\leq \|T^{n+1}y - T^n y\| + \frac{1-K^n}{1-K}\|Ty - y\| \text{ (by (26.23))} && (26.27)\\
&= \|T^n Ty - T^n y\| + \frac{1-K^n}{1-K}\|Ty - y\| && (26.28)\\
&\leq K^n\|Ty - y\| + \frac{1-K^n}{1-K}\|Ty - y\| \text{ (because } T \text{ is a contraction)} && (26.29)\\
&= \frac{(1-K)K^n + (1-K^n)}{1-K}\|Ty - y\| && (26.30)\\
&= \frac{1-K^{n+1}}{1-K}\|Ty - y\| && (26.31)
\end{aligned}$$

which proves the conjecture for $n + 1$. □

Definition 26.9. Let $\mathcal{V}$ be a vector space let T be an operator on $\mathcal{V}$. Then we say y is a **fixed point** of T if $Ty = y$.

Note that in the vector space of functions, since the vectors are functions, the fixed point is a function.

Theorem 26.10. Contraction Mapping Theorem[1] Let T be a contraction on a normed vector space V. Then T has a unique fixed point $u \in \mathcal{V}$ such that $Tu = u$. Furthermore, any sequence of vectors $v_1, v_2, \dots$ defined by $v_k = Tv_{k-1}$ converges to the unique fixed point $Tu = u$. We denote this by $v_k \to u$.

Proof. [2] Let $\epsilon > 0$ be given and let $v \in \mathcal{V}$.

[1] The contraction mapping theorem is sometimes called the Banach Fixed Point Theorem.

[2] The proof follows "Proof of Banach Fixed Point Theorem," *Encyclopedia of Mathematics* (Volume 2, 54A20:2034), PlanetMath.org.

Since $K^n/(1-K) \to 0$ as $n \to \infty$ (because T is a contraction, $K < 1$), given any $v \in \mathcal{V}$, it is possible to choose an integer N such that

$$\frac{K^n\|Tv - v\|}{1-K} < \epsilon \tag{26.32}$$

for all $n > N$. Pick any such integer N.

Choose any two integers $m \geq n \geq N$, and define the sequence

$$\left.\begin{array}{l} v_0 = v \\ v_1 = Tv \\ v_2 = Tv_1 \\ \vdots \\ v_n = Tv_{n-1} \\ \vdots \end{array}\right\} \tag{26.33}$$

Then since T is a contraction,

$$\begin{aligned} \|v_m - v_n\| &= \|T^m v - T^n v\| && (26.34) \\ &= \|T^n T^{m-n} v - T^n v\| && (26.35) \\ &\leq K^n \|T^{m-n} v - v\| && (26.36) \end{aligned}$$

From Lemma 26.8 we have

$$\begin{aligned} \|v_m - v_n\| &\leq K^n \frac{1-K^{m-n}}{1-K} \|Tv - v\| && (26.37) \\ &= \frac{K^n - K^m}{1-K} \|Tv - v\| && (26.38) \\ &\leq \frac{K^n}{1-K} \|Tv - v\| < \epsilon && (26.39) \end{aligned}$$

Therefore v_n is a Cauchy sequence, and every Cauchy sequence on a complete normed vector space converges. Hence $v_n \to u$ for some $u \in \mathcal{V}$.

Either u is a fixed point of T or it is not a fixed point of T.

Suppose that u is not a fixed point of T. Then $Tu \neq u$ and hence there exists some $\delta > 0$ such that

$$\|Tu - u\| > \delta \tag{26.40}$$

On the other hand, because $v_n \to u$, there exists an integer N such that for all $n > N$,

$$\|v_n - u\| < \delta/2 \tag{26.41}$$

Hence

$$\begin{aligned}
\|Tu - u\| &\leq \|Tu - v_{n+1}\| + \|v_{n+1} - u\| && (26.42)\\
&= \|Tu - Tv_n\| + \|u - v_{n+1}\| && (26.43)\\
&\leq K\|u - v_n\| + \|u - v_{n+1}\| \qquad \text{(because } T \text{ is a contraction)} && (26.44)\\
&\leq \|u - v_n\| + \|u - v_{n+1}\| \qquad \text{(because } K < 1\text{)} && (26.45)\\
&= 2\|u - v_n\| && (26.46)\\
&< \delta && (26.47)
\end{aligned}$$

This is a contradiction. Hence u must be a fixed point of T.

To prove uniqueness, suppose that there is another fixed point $w \neq u$. Then $\|w - u\| > 0$ (otherwise they are equal). But

$$\|u - w\| = \|Tu - Tw\| \leq K\|u - w\| < \|u - w\| \tag{26.48}$$

which is impossible and hence and contradiction.

Thus u is the unique fixed point of T. □

Theorem 26.11. Fundamental Existence Theorem. Let $D \in \mathbb{R}^2$ be convex and suppose that f is continuously differentiable on D. Then the initial value problem

$$y' = f(t, y), \quad y(t_0) = y_0 \tag{26.49}$$

has a unique solution $\phi(t)$ in the sense that $\phi'(t) = f(t, \phi(y)), \phi(t_0) = y_0$.

Proof. We begin by observing that ϕ is a solution of equation 26.49 if and only if it is a solution of

$$\phi(t) = y_0 + \int_{t_0}^{t} f(x, \phi(x))dx \tag{26.50}$$

Our goal will be to prove 26.50.

Let $\mathcal{V}$ be the set of all continuous integrable functions on an interval (a, b) that contains t_0. Then $\mathcal{V}$ is a complete normed vector space with the sup-norm as norm, as we have already seen. mn

Define the linear operator T on $\mathcal{V}$ by

$$T(\phi) = y_0 + \int_{t_0}^{t} f(s, \phi(s))ds \tag{26.51}$$

for any $\phi \in \mathcal{V}$.

Let g, h be functions in $\mathcal{V}$.

$$\|Tg - Th\|_\infty = \sup_{a \le t \le b} |Tg - Th| \tag{26.52}$$

$$= \sup_{a \le t \le b} \left| y_0 + \int_{t_0}^{t} f(x, g(x))dx - y_0 - \int_{t_0}^{t} f(x, h(x))dx \right| \tag{26.53}$$

$$= \sup_{a \le t \le b} \left| \int_{t_0}^{t} \left[f(x, g(x)) - f(x, h(x)) \right] dx \right| \tag{26.54}$$

Since f is continuously differentiable it is differentiable and its derivative is continuous. Thus the derivative is bounded (otherwise it could not be continuous on all of (a, b)). Therefore by theorem 12.3, it is Lipshitz in its second argument. Consequently there is some $K \in \mathbb{R}$ such that

$$\|Tg - Th\|_\infty \le L \sup_{a \le t \le b} \int_{t_0}^{t} |g(x) - h(x)|\, dx \tag{26.55}$$

$$\le K(t - t_0) \sup_{a \le t \le b} |g(x)) - h(x)| \tag{26.56}$$

$$\le K(b - a) \sup_{a \le t \le b} |g(x)) - h(x)| \tag{26.57}$$

$$\le K(b - a) \|g - h\| \tag{26.58}$$

Since K is fixed, so long as the interval (a, b) is larger than $1/K$ we have

$$\|Tg - Th\|_\infty \le K' \|g - h\|_\infty \tag{26.59}$$

where

$$K' = K(b - a) < 1 \tag{26.60}$$

Thus T is a contraction. By the contraction mapping theorem it has a fixed point; call this point ϕ. Equation 26.50 follows immediately. □

This theorem means that higher order initial value problems also have unique solutions. Why is this? Because any higher order differential equation can be converted to a system of equations, as in the following example.

Example 26.2. Convert initial value problem

$$\left.\begin{aligned} y'' + 4t^3 y' + y^3 &= \sin(t) \\ y(0) &= 1 \\ y'(0) &= 1 \end{aligned}\right\} \tag{26.61}$$

to a system.

We create a system by defining the variables

$$\begin{aligned} x_1 &= y \\ x_2 &= y' \end{aligned} \tag{26.62}$$

Then the differential equation becomes

$$x_2' + 4t^3 x_2 + x_1^3 = \sin t \tag{26.63}$$

which we can rewrite as

$$x_2' = \sin t - x_1^3 - 4t^3 x_2 \tag{26.64}$$

We then define functions f, and g,

$$\begin{aligned} f(x_1, x_2) &= \sin t - x_1^3 - 4t^3 x_2 \\ g(x_1, x_2) &= x_2 \end{aligned} \tag{26.65}$$

so that our system can be written as

$$\begin{aligned} x_1' &= f(x_1, x_2) \\ x_2' &= g(x_1, x_2) \end{aligned} \tag{26.66}$$

with initial condition

$$\begin{aligned} x_1(0) &= 1 \\ x_2(0) &= 1 \end{aligned} \tag{26.67}$$

It is common to define a vector $\mathbf{x} = (x_1, x_2)$ and a vector function

$$\mathbf{F}(\mathbf{x}) = (f(x_1, x_2), g(x_1, x_2)) \tag{26.68}$$

Then we have a vector initial value problem

$$\left.\begin{aligned} &\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}) = (\sin t - x_1^3 - 4t^3 x_2, x_2) \\ &\mathbf{x}(0) = (1, 1) \end{aligned}\right\} \tag{26.69}$$

Since the set of all differentiable functions on $\mathbb{R}^2$ is a vector space, our theorem on vector spaces applies. Even though we proved theorem (26.50) for first order equations every step in the proof still works when y and f become vectors. On any closed rectangle surrounding the initial condition $\mathbf{F}$ and $\partial \mathbf{F}/\partial x_i$ is bounded, continuous, and differentiable. So there is a unique solution to this initial value problem. $\square$

Lesson 27

Higher Order Linear Equations

Constant Coefficients and the Linear Operator

Generalizing the previous sections we can write the general nth-order linear equation with constant coefficients as

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(t) \tag{27.1}$$

The corresponding characteristic polynomial is

$$\begin{aligned} P_n(r) &= a_n r^n + a_{n-1} r^{n-1} + \cdots + a_0 && (27.2)\\ &= a_n (r - r_1)(r - r_2)\cdots(r - r_n) && (27.3)\\ &= 0 && (27.4) \end{aligned}$$

and the corresponding n-th order linear operator is

$$\begin{aligned} L_n &= a_n D^n + a_{n-1} D^{n-1} + \cdots + a_0 && (27.5)\\ &= a_n (D - r_1)(D - r_2)\cdots(D - r_n) && (27.6) \end{aligned}$$

where $a_0, ..., a_n \in \mathbb{R}$ are constants and $r_1, r_2, ..., r_n \in \mathbb{C}$ are the roots of $P_n(r) = 0$ (some or all of which may be repeated). The corresponding

initial value problem is

$$\left.\begin{aligned} L_n y &= f(t) \\ y(t_0) &= y_0 \\ y'(t_0) &= y_1 \\ &\vdots \\ y^{(n)}(t_0) &= y_n \end{aligned}\right\} \tag{27.7}$$

where $y_0, ..., y_n \in \mathbb{R}$ are constants.

While the roots of $P_n(r) = 0$ may be complex we will restrict the coefficients and initial conditions to be real. The linear operator has the following properties:

$$L_n(y) = P_n(D)y \tag{27.8}$$

$$L_n(e^{rt}) = e^{rt}P_n(r) \tag{27.9}$$

$$L_n(e^{rt}y) = e^{rt}P_n(D+r)y \tag{27.10}$$

Equations (27.8) and (27.9) are straightforward; (27.10) can be proven by induction.

Proof of (27.10) by induction (inductive step only).* Assume $L_n(e^{rt}y) = e^{rt}P_n(D+a)$. Then

$$L_{n+1}(e^{rt}y) = a_{n+1}(D-r_1)\cdots(D-r_{n+1})e^{rt}y \tag{27.11}$$

$$= (D-r_1)z \tag{27.12}$$

where

$$z = a_{n+1}(D-r_2)(D-r_3)\cdots(D-r_{n+1})(e^{rt}y) \tag{27.13}$$

Since (27.12) is an nth order equation, the inductive hypothesis holds for it, namely, that

$$z = e^{rt}a_{n+1}(D+r-r_2)(D+r-r_3)\cdots(D+r-r_{n+1})y = e^{rt}u \tag{27.14}$$

where

$$u = a_{n+1}(D+r-r_2)\cdots(D+r-r_{n+1})y \tag{27.15}$$

Substituting,

$$\begin{aligned} L_{n+1}(e^{rt}y) &= (D-r_1)z = (D-r_1)e^{rt}u \\ &= re^{rt}u + e^{rt}u' - r_1e^{rt}u \\ &= e^{rt}(D+r-r_1)u \end{aligned} \tag{27.16}$$

Substituting back for u from (27.15) and applying the definition of $P_{n+1}(x)$

$$\begin{aligned} L_{n+1}(e^{rt}y) &= e^{rt}a_n(D+r-r_1)(D+r-r_2)\cdots(D+r-r_{n+1})y \\ &= e^{rt}P_{n+1}(D+r)y \end{aligned} \tag{27.17}$$

which proves the assertion for $n+1$, completing the inductive proof. $\square$.

The general solution to (27.1) is

$$y = y_H + y_P \tag{27.18}$$

where

$$y_H = C_1 y_{H,1} + C_2 y_{H,2} + \cdots + C_n y_{H,n} \tag{27.19}$$

and the $y_{H,i}$ are linearly independent solutions of the homogeneous equation $L_n y = 0$. If L is n-th order then there will be n linearly independent solutions; taken together, any set of n linearly independent solutions are called a **fundamental set of solutions**. Note that the set is not unique, because if y is a solution the differential equation then so is cy for any constant c.

Superposition and Subtraction

As before with second order equations, we have a principle of superposition and a subtraction principle.

Theorem 27.1. (Principle of Superposition.) If $u(t)$ and $v(t)$ are any two solutions of $L_n y = 0$ then any linear combination $w(t) = c_1 u(t) + c_2 v(t)$ is also a solution of $L_n y = 0$.

Theorem 27.2. (Subtraction Principle)If $u(t)$ and $v(t)$ are any solutions to $L_n y = f(t)$ then $w(t) = u(t) - v(t)$ is a solution to the homogeneous equation $L_n y = 0$.

The Homogeneous Equation

The solutions we found for second order equations generalize to higher order equations. In fact, the solutions to the homogeneous equation are the functions e^{rt}, te^{rt},..., $t^{k-1}e^{rt}$ where each r is a root of $P_n(r)$ with multiplicity k, and these are the only solutions to the homogeneous equation, as we prove in the following two theorems.

Theorem 27.3. Let r be a root of $P_n(r)$ with multiplicity k. Then $e^{rt}, te^{rt}, ..., t^{k-1}e^{rt}$ are solutions to the homogeneous equation $L_n y = 0$.

Proof. Since r is a root of $P_n(r)$, then $P_n(r) = 0$. Hence

$$L_n e^{rt} = e^{rt} P_n(r) = 0 \tag{27.20}$$

Suppose that r has multiplicity k. Renumber the roots $r_1, r_2, ..., r_n$ as $r_1, r_2, ..., r_{n-k}, \underbrace{r, ..., r}_{k\ times}$. Then

$$P_n(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_{n-k})(x - r)^k \tag{27.21}$$

Let $s \in \{0, 1, 2, ..., k-1\}$ so that $s < k$ is an integer. Then by the polynomial shift property (27.10) and equation (27.21)

$$L_n(e^{rt}t^s) \quad = \quad e^{rt} P_n(D + r)t^s \tag{27.22}$$

because $D^k(t^s) = 0$ for any integer $s < k$. □

Theorem 27.4. Suppose that the roots of $P_n(r) = a_n r^n + a_{n-1}r^{n-1} + \cdots + a_0$ are $r_1, ..., r_k$ with multiplicities $m_1, m_2, ..., m_k$ (where $m_1 + m_2 + \cdots + m_k = n$). Then the general solution of $L_n y = 0$ is

$$y_H = \sum_{i=1}^{k} e^{r_i t} \sum_{j=0}^{m_i - 1} C_{ij} t^j \tag{27.23}$$

Before we prove theorem 27.3 will consider several examples

Example 27.1. Find the general solution of $y''' - 3y'' + 3y' - y = 0$ The characteristic equation is

$$r^3 - 3r^2 + 3r - 1 = (r - 1)^3 = 0 \tag{27.24}$$

which has a single root $r = 1$ with multiplicity 3. The general solution is therefore

$$y = (C_1 + C_2 t + C_3 t^3)e^t \quad \square \tag{27.25}$$

Example 27.2. Find the general solution of $y^{(4)} - 2y'' + y = 0$

The characteristic equation is

$$0 = r^4 - 2r^2 + r = (r^2-1)^2 = (r-1)^2(r+1)^2 \tag{27.26}$$

The roots are 1 and -1, each with multiplicity 2. Thus the solution is

$$y = e^t(C_1 + C_2 t) + e^{-t}(C_3 + C_4 t) \quad \square \tag{27.27}$$

Example 27.3. Find the general solution of $y^{(4)} - y = 0$

The characteristic equation is

$$0 = r^4 - 1 = (r^2-1)(r^2+1) = (r-1)(r+1)(r-i)(r+i) \tag{27.28}$$

There are four distinct roots $r = \pm 1, \pm i$. The real roots give solutions $e^{\pm t}$; the complex roots $r = 0 \pm i$ give terms $\sin t$ and $\cos t$. Hence the general solution is

$$y = C_1 e^t + C_2 e^{-t} + C_3 \cos t + C_4 \sin t \quad \square \tag{27.29}$$

Example 27.4. Find the general solution of $y^{(4)} + y = 0$

The characteristic equation is $r^4 + 1 = 0$.

Therefore $r = (-1)^{1/4}$. By Euler's equation,

$$-1 = e^{i\pi} = e^{i(\pi + 2k\pi)} \tag{27.30}$$

hence the four fourth-roots are

$$\begin{aligned}(-1)^{1/4} &= \exp\left[\frac{i(\pi+2k\pi)}{4}\right] &(27.31)\\ &= \cos\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{k\pi}{2}\right),\ k = 0,1,2,3 &(27.32)\end{aligned}$$

Therefore

$$r_i = \pm\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(\pm 1 \pm i) = \frac{1}{\sqrt{2}}(\pm 1 \pm i) \tag{27.33}$$

and the general solution of the differential equation is

$$\begin{aligned}y = e^{t/\sqrt{2}}&\left[C_1 \cos\frac{t}{\sqrt{2}} + C_2 \sin\frac{t}{\sqrt{2}}\right]\\ &+ e^{-t/\sqrt{2}}\left[C_3\cos\frac{t}{\sqrt{2}} + C_4 \sin\frac{t}{\sqrt{2}}\right] \quad \square\end{aligned} \tag{27.34}$$

To prove theorem 27.3[1] we will need a generalization of the fundamental identity. We first define a the following **norm of a function**:

$$\|f(t)\|^2 = \sum_{i=0}^{n-1} \left|f^{(i)}(t)\right|^2 \tag{27.35}$$

$$= |f(t)| + |f'(t)| + \cdots + \left|f^{(n-1)}(t)\right|^2 \tag{27.36}$$

That $\|\cdots\|$ defines a norm follows from the fact that for any two functions $f(t)$ and $g(t)$, and for and constant c, the following four properties are true for all t:

Properties of a Norm of a Function:

1. $\|f(t)\| \geq 0$
2. $\|f(t)\| = 0 \ \Leftrightarrow \ f(t) = 0$
3. $\|f(t) + g(t)\| \leq \|f(t)\| + \|g(t)\|$
4. $\|cf(t)\| \leq |c| \, \|f(t)\|$

For comparison to the norm of a vector space, see definition 15.2 which is equivalent to this for the vector space of functions.

Lemma 27.5. (Fundamental Identity for nth order equations) Let $\phi(t)$ be a solution of

$$L_n y = 0, \ y(t_0) = y_0, ..., y^{(n-1)}(t_0) = y_n \tag{27.37}$$

then

$$\|\phi(t_0)\| \, e^{-K\|t-t_0\|} \leq \|\phi(t)\| \leq \|\phi(t_0)\| \, e^{K\|t-t_0\|} \tag{27.38}$$

for all t.

Lemma 27.6. $2\,|a|\,|b| \leq |a|^2 + |b|^2$

Proof. $(|a| + |b|)^2 = |a|^2 + |b|^2 - 2\,|a|\,|b| \geq 0.$ □

Proof. (of Lemma 27.5.) We can assume that $a_n \neq 0$; otherwise this would not be an nth-order equation. Further, we will assume that $a_n = 1$; otherwise, redefine L_n by division through by a_n.

Let

$$u(t) = \|\phi(t)\|^2 = \sum_{i=0}^{n-1} \left|\phi^{(i)}(t)\right|^2 = \sum_{i=0}^{n-1} \phi^{(i)}(t)\phi^{(i)*}(t) \tag{27.39}$$

[1]This material is somewhat more abstract and the reader may wish to skip ahead to the examples following the proofs.

Differentiating,

$$u'(t) = \frac{d}{dt}\sum_{i=0}^{n-1}\left|\phi^{(i)}(t)\right|^2 = \frac{d}{dt}\sum_{i=0}^{n-1}\phi^{(i)}(t)\phi^{(i)*}(t) \tag{27.40}$$

Taking the absolute value and applying the triangle inequality twice

$$|u'(t)| \leq \sum_{i=0}^{n-1}\left|\phi^{(i+1)}(t)\phi^{(i)*}(t) + \phi^{(i)}(t)\phi^{(i+1)*}(t)\right| \tag{27.41}$$

Since $|a| = |a^*|$

$$|u'(t)| \leq \sum_{i=0}^{n-1} 2\left|\phi^{(i+1)}(t)\right|\left|\phi^{(i)}(t)\right| \tag{27.42}$$

Isolating the highest order term

$$|u'(t)| \leq \left\{\sum_{i=0}^{n-2} 2\left|\phi^{(i+1)}(t)\right|\left|\phi^{(i)}(t)\right|\right\} + 2\left|\phi^{(n)}(t)\right|\left|\phi^{(n-1)}(t)\right| \tag{27.43}$$

Since $L_n\phi(t) = 0$ and $a_n = 1$,

$$\left|\phi^{(n)}(t)\right| = \left|-\sum_{i=0}^{n-1} a_i\phi^{(i)}(t)\right| \leq \sum_{i=0}^{n-1}\left|a_i\phi^{(i)}(t)\right| = \sum_{i=0}^{n-1}|a_i|\left|\phi^{(i)}(t)\right| \tag{27.44}$$

Combining the last two results,

$$|u'(t)| \leq \left\{\sum_{i=0}^{n-2} 2\left|\phi^{(i+1)}(t)\right|\left|\phi^{(i)}(t)\right|\right\} + 2\left|\sum_{i=0}^{n-1}|a_i|\left|\phi^{(i)}(t)\right|\right|\left|\phi^{(n-1)}(t)\right| \tag{27.45}$$

By lemma 2,

$$2\left|\phi^{(i+1)}\right|\left|\phi^{(i)}\right| \leq \left|\phi^{(i+1)}\right|^2 + \left|\phi^{(i)}\right|^2 \tag{27.46}$$

Hence

$$\begin{aligned} |u'(t)| \leq &\left\{\sum_{i=0}^{n-2}\left(\left|\phi^{(i+1)}(t)\right|^2 + \left|\phi^{(i)}(t)\right|^2\right)\right\} \\ &+2\left|\phi^{(n-1)}(t)\right|\sum_{i=0}^{n-1}|a_i|\left|\phi^{(i)}(t)\right| \end{aligned} \tag{27.47}$$

By a change of index in the first term (let $j = i + 1$)

$$\sum_{i=0}^{n-2}\left|\phi^{(i+1)}\right|^2 = \sum_{j=1}^{n-1}\left|\phi^{(j)}(t)\right|^2 \tag{27.48}$$

so that

$$|u'(t)| \le \sum_{i=1}^{n-1} \left|\phi^{(i)}(t)\right|^2 + \sum_{i=0}^{n-2} \left|\phi^{(i)}(t)\right|^2 + 2\left|\phi^{(n-1)}(t)\right| \sum_{i=0}^{n-1} |a_i| \left|\phi^{(i)}(t)\right| \tag{27.49}$$

Since

$$\sum_{i=1}^{n-1} |c_i| \le \sum_{i=0}^{n-1} |c_i| \tag{27.50}$$

and

$$\sum_{i=0}^{n-2} |c_i| \le \sum_{i=0}^{n-1} |c_i| \tag{27.51}$$

for any set of numbers c_i, this becomes

$$\begin{aligned} |u'(t)| \le \sum_{i=0}^{n-1} \left|\phi^{(i)}(t)\right|^2 + \sum_{i=0}^{n-1} \left|\phi^{(i)}(t)\right|^2 \\ +2\left|\phi^{(n-1)}(t)\right| \sum_{i=0}^{n-1} |a_i| \left|\phi^{(i)}(t)\right| \end{aligned} \tag{27.52}$$

From equation (27.39),

$$|u'(t)| \le 2u(t) + 2\left|\phi^{(n-1)}(t)\right| \sum_{i=0}^{n-1} |a_i| \left|\phi^{(i)}(t)\right| \tag{27.53}$$

From lemma 1,

$$\det M \ne 0 \tag{27.54}$$

Therefore,

$$|u'(t)| \le 2u(t) + \sum_{i=0}^{n-1} |a_i|\, 2u(t) \tag{27.55}$$

$$= 2u(t)\left[1 + \sum_{i=0}^{n-1} |a_i|\right] = 2Ku(t) \tag{27.56}$$

Hence

$$-2K \le \frac{u'(t)}{u(t)} \le 2K \tag{27.57}$$

Let $u(t_0) = u_0$ and integrate from t_0 to t

$$-2K\int_{t_o}^{t} ds \le \int_{t_o}^{t} \frac{u'(s)}{u(s)} ds \le 2K\int_{t_o}^{t} ds \tag{27.58}$$

Integrating,

$$-2K\,|t-t_0| \le \ln|u(t)/u(t_0)| \le 2K\,|t-t_0| \tag{27.59}$$

Exponentiating,

$$|u_0(t)|\,e^{-2K|t-t_0|} \le |u(t)| \le |u_0(t)|\,e^{2K|t-t_0|} \tag{27.60}$$

By equation (27.39) this last statement is equivalent to the desired result, which is the fundamental inequality. □

Proof. (Theorem 27.3) By the previous theorem, each term in the sum is a solution of $L_n y = 0$, and hence by the superposition principle, (27.23) is a solution.

To prove that it is the general solution we must show that every solution of $L_n y = 0$ has the form (27.23).

Suppose that $u(t)$ is a solution that does not have the form given by (27.23), i.e., it is not a linear combination of the $y_{ij} = e^{r_i t}t^j$.

Renumber the y_{ij} to have a single index $y_1, ..., y_n$, and let $u_0 = u(t_0), u_1 = u'(t_0), ..., u_{n-1} = u^{(n-1)}(t_o)$. Then $u(t)$ is a solution of some initial value problem

$$L_n y = 0,\ y(t_0) = u_0, y'(t_0) = u_1, ..., y^{(n-1)}(t_0) = u_{n-1} \tag{27.61}$$

and by uniqueness it must be the only solution of (27.61). Let

$$v = c_1y_1 + c_2y_2 + \cdots + c_ny_n \tag{27.62}$$

Differentiating n times,

$$v' = c_1y_1' + c_2y_2' + \cdots + c_ny_n' \tag{27.63}$$

We ask whether there is a solution $c_1, c_2, ..., c_n$ to the system

$$v(t_0) = c_1y_1(t_0) + c_2y_2(t_0) + \cdots + c_ny_n(t_0) = u_0 \tag{27.64}$$

If the matrix

$$M = \begin{pmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & & y_n'(t_0) \\ \vdots & & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & & y_n^{(n-1)}(t_0) \end{pmatrix} \tag{27.65}$$

is invertible, then a solution of (27.64) is

$$c = M^{-1}u_0 \tag{27.66}$$

where $c = (c_1 \quad c_2 \quad \cdots \quad c_n)^{\mathrm{T}}$ and $u_0 = (u_0 \quad u_1 \quad \cdots \quad u_{n-1})^{\mathrm{T}}$. But $\mathbf{M}$ is invertible if and only if $\det M \neq 0$. We will prove this by contradiction.

Suppose that $\det M = 0$. Then $Mc = 0$ for some non-zero vector $\mathbf{c}$, i.e,

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & & y_n'(t_0) \\ \vdots & & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & & y_n^{(n-1)}(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = 0 \tag{27.67}$$

and line by line,

$$v^{(j)}(t_0) = \sum_{i=1}^{n} c_i y_i^{(j)}(t_0) = 0, \quad j = 0, 1, ..., n-1 \tag{27.68}$$

Using the norm defined by (27.35),

$$\|v(t_0)\|^2 = \sum_{i=0}^{n-1} \left| v^{(j)}(t_0) \right|^2 = 0 \tag{27.69}$$

By the fundamental inequality, since $v(t)$ is a solution,

$$\|v(t_0)\| e^{-K|t-t_0|} \leq \|v(t)\| \leq \|v(t_0)\| e^{K|t-t_0|} \tag{27.70}$$

Hence $\|v(t)\| = 0$, which means $v(t) = 0$ for all t. Since all of the $y_i(t)$ are linearly independent, this means that all of the $c_i = 0$, i.e., $c = 0$. But this contradicts (27.67), so it must be true that $\det M \neq 0$.

Since $\det M \neq 0$, the solution given by (27.66) exists. Thus $v(t)$, which exists as a linear combination of the y_i is a solution of the same initial value problem as $u(t)$. Thus $v(t)$ and $u(t)$ must be identical by uniqueness, and our assumptions that $u(t)$ was not a linear combination of the y_i is contradicted. This must mean that no such solution exists, and every solution of $L_n y = 0$ must be a linear combination of the y_i. $\square$

The Particular Solution

The particular solution can be found by the method of undetermined coefficients or annihilators, or by a generalization of the expression that gives

a closed form expression for a particular solution to $L_n y = f(t)$. Such an expression can be found by factoring the differential equation as

$$\begin{aligned} L_n y &= a_n(D-r_1)(D-r_2)\cdots(D-r_n)y && (27.71)\\ &= a_n(D-r_1)z = f(t) && (27.72)\end{aligned}$$

where $z=(D-r_2)\cdots(D-r_n)y$. Then

$$z' - r_1 z = (1/a_n)f(t) \tag{27.73}$$

An integrating factor is $\mu = e^{-r_1 t}$, so that

$$\begin{aligned} (D-r_2)\cdots(D-r_n)y &= z && (27.74)\\ &= \frac{1}{a_n}e^{r_1 t}\int_t e^{-r_1 s_1} f(s_1)ds_1 = f_1(t) && (27.75)\end{aligned}$$

where the last expression on the right hand side of (27.74) is taken as the definition of $f_1(t)$. We have ignored the constants of integration because they will give us the homogeneous solutions.

Defining a new $z = (D-r_3)\cdots(D-r_n)y$ gives

$$z' - r_2 z = f_1(t) \tag{27.76}$$

An integrating factor for (27.76) is $\mu = e^{-r_2 t}$, so that

$$(D-r_3)\cdots(D-r_n)y = z = e^{r_2 t}\int_t e^{-r_2 s_2} f_1(s_2)ds_2 = f_2(t) \tag{27.77}$$

where the expression on the right hand side of (27.77) is taken as the definition of $f_2(t)$. Substituting for $f_1(s_2)$ from (27.74) into (27.77) gives

$$(D-r_3)\cdots(D-r_n)y = e^{r_2 t}\int_t e^{-r_2 s_2}\frac{1}{a_n}e^{r_1 s_2}\int_{s_2} e^{-r_1 s_1} f(s_1)ds_1 ds_2 \tag{27.78}$$

Repeating this procedure n times until we have exhausted all of the roots,

$$\boxed{\begin{aligned} y_P = \frac{e^{r_n t}}{a_n}\int_t e^{(r_{n-1}-r_n)s_n}\int_{s_n} e^{(r_{n-2}-r_{n-1})s_{n-1}}\cdots\\ \cdots\int_{s_3} e^{(r_1-r_2)s_2}\int_{s_2} e^{-r_1 s_1} f(s_1)ds_1\cdots ds_n\end{aligned}} \tag{27.79}$$

□

Example 27.5. Find the general solution to $y''' + y'' - 6y' = e^t$.

The characteristic equation is

$$0 = r^3 + r^2 - 6r = r(r-2)(r+3) \tag{27.80}$$

which has roots $r_1 = 0$, $r_2 = 2$, and $r_3 = -3$. The general solution to the homogeneous equation is

$$y_H = C_1 + C_2 e^{2t} + C_3 e^{-3t} \tag{27.81}$$

From (27.79), a particular solution is

$$\begin{aligned}
y_P &= e^{r_3 t} \int_t e^{(r_2 - r_3)s_3} \int_{s_3} e^{(r_1 - r_2)s_2} \int_{s_2} e^{-r_1 s_1} f(s_1) ds_1 ds_2 ds_3 && (27.82)\\
&= e^{-3t} \int_t e^{5s_3} \int_{s_3} e^{-2s_2} \int_{s_2} e^{s_1} ds_1 ds_2 ds_3 && (27.83)\\
&= e^{-3t} \int_t e^{5s_3} \int_{s_3} e^{-2s_2} e^{s_2} ds_2 ds_3 && (27.84)\\
&= e^{-3t} \int_t e^{5s_3} \int_{s_3} e^{-s_2} ds_2 ds_3 && (27.85)\\
&= -e^{-3t} \int_t e^{5s_3} e^{-s_3} ds_3 && (27.86)\\
&= -e^{-3t} \int_t e^{4s_3} ds_3 && (27.87)\\
&= -\frac{1}{4} e^{-3t} e^{4t} && (27.88)\\
&= -\frac{1}{4} e^{t} && (27.89)
\end{aligned}$$

Hence the general solution is

$$y = y_P + y_H = -\frac{1}{4} e^t + C_1 + C_2 e^{2t} + C_3 e^{-3t} \tag{27.90}$$

In general it is easier to use undetermined coefficients to determine y_P if a good guess for its form is known, rather than keeping track of the integrals in (27.79). Failing that the bookkeeping still tends to be easier if we reproduce the derivation of (27.79) by factoring the equation one root at a time than it is to use (27.79) directly. In general the best "guess" for the form of a particular solution is the same for higher order equations as it is for second order equations. For example, in the previous example we would have looked for a solution of the form $y_P = ce^t$.

Example 27.6. Find the general solution of $y^{(4)} - 5y'' + 4y = t$
The characteristic equation is $0 = r^4 - 5r^2 + 4 = (r^2 - 1)(r^2 - 4)$ so the roots are 1, -1, 2, and -2, and the homogeneous solution is

$$y_H = C_1 e^t + C_2 e^{-t} + C_3 e^{2t} + C_4 e^{-2t} \tag{27.91}$$

Using the factorization method: we write the differential equation as

$$(D-1)(D+1)(D-2)(D+2)y = (D-1)z = t \tag{27.92}$$

where $z = (D+1)(D-2)(D+2)y$. Then $z' - z = t$. An integrating factor is e^{-t}, so that

$$z = e^t \int te^{-t} dt = e^t \left[-(t+1)e^{-t}\right] = -t - 1 \tag{27.93}$$

where we have ignored the constant of integration because we know that they will lead to the homogeneous solutions. Therefore

$$z = (D+1)(D-2)(D+2)y = (D+1)w = -t - 1 \tag{27.94}$$

where $w = (D-2)(D+2)y$. Equation (27.94) is equivalent to $w' + w = -t - 1$, so that

$$\begin{aligned} w &= -e^{-t}\left[\int e^t(t+1)dt\right] && (27.95)\\ &= -e^{-t}\left[\int te^t dt + \int e^t dt\right] && (27.96)\\ &= -e^{-t}\left[(t-1)e^t + e^t\right] && (27.97)\\ &= -t && (27.98)\end{aligned}$$

Therefore

$$\begin{aligned} w &= (D-2)(D+2)y && (27.99)\\ &= (D-2)u = -t && (27.100)\end{aligned}$$

where $u = (D+2)y = y' + 2y$. Equation (27.99) is $u' - 2u = -t$. An integrating factor is e^{-2t} and the solution for u is

$$u = -e^{2t} \int te^{-2t} dt = -e^{2t}\left[-\frac{1}{2}t - \frac{1}{4}\right] e^{-2t} = \frac{1}{2}t + \frac{1}{4} \tag{27.101}$$

Therefore

$$y' + 2y = \frac{1}{4}(2t+1) \tag{27.102}$$

An integrating factor is e^{2t} so that

$$\begin{aligned} y &= \frac{1}{4}e^{-2t}\int e^{2t}(2t+1)dt && (27.103)\\ &= \frac{1}{4}e^{-2t}\left[2\int te^{2t}dt + \int e^{2t}dt\right] && (27.104)\\ &= \frac{1}{4}e^{-2t}\left[2\left(\frac{t}{2}-\frac{1}{4}\right)e^{2t} + \frac{1}{2}e^{2t}\right] && (27.105)\\ &= \frac{t}{4} && (27.106)\end{aligned}$$

Therefore $y_P = t/4$.

Alternatively, using the method of undetermined coefficients, we try $y_P = ct$ in the differential equation $y^{(4)} - 5y'' + 4y = t$. Since $y' = c$ and $y'' = y^{(4)} = 0$ we find $4ct = t$ or $c = 1/4$, again giving $y_P = t/4$.

Hence the general solution is

$$y = y_P + y_H = \frac{1}{4}t + C_1e^t + C_2e^{-t} + C_3e^{2t} + C_4e^{-2t} \quad \square \qquad (27.107)$$

We also have an addition theorem for higher order equations.

Theorem 27.7. If $y_{P,i}, \;\; i = 1, 2, ..., k$ are particular solutions of $L_n y_{P,i} = f_i(t)$ then

$$y_P = y_{P,1} + y_{P,2} + \cdots + y_{P,k} \qquad (27.108)$$

is a particular solution of

$$L_n y = f_1(t) + f_2(t) + \cdots + f_k(t) \qquad (27.109)$$

Example 27.7. Find the general solution of $y''' - 4y' = t + 3\cos t + e^{-2t}$.

The general solution is

$$y = y_H + y_1 + y_2 + y_3 \qquad (27.110)$$

where

$$\begin{aligned} y_H''' - 4y_H' &= 0 && (27.111)\\ y_1''' - 4y_1' &= t && (27.112)\\ y_2''' - 4y_2' &= 3\cos t && (27.113)\\ y_3''' - 4y_3' &= e^{-2t} && (27.114)\end{aligned}$$

The characteristic equation is

$$0 = r^3 - 4r = r(r^2 - 4) = r(r-2)(r+2) \tag{27.115}$$

Since the roots are $r = 0, \pm 2$, the solution of the homogeneous equation is

$$y_H = C_1 + C_2 e^{2t} + C_3 e^{-2t} \tag{27.116}$$

To find y_1, our first inclination would be to try $y = At + B$. But e^{0t} is a solution of the homogeneous equation so we try $y + 1 = t^k(at + b)$ with $k = 1$.

$$y_1 = t^k(a + bt)e^{0t} = at + bt^2 \tag{27.117}$$

Differentiating three times

$$y_1' = a + 2bt \tag{27.118}$$

$$y_1'' = 2b \tag{27.119}$$

$$y_1''' = 0 \tag{27.120}$$

Substituting into the differential equation gives

$$t = y_1''' - 4y_1' = 0 - 4(a + 2bt) = -4a - 8bt \tag{27.121}$$

This must hold for all t, so equating like coefficients of t gives us $a = 0$ and $b = -1/8$ so that the first particular solution is

$$y_1 = -\frac{1}{8}t^2 \tag{27.122}$$

For y_2, since $r = \pm i$ are not roots of the characteristic equation we try

$$y_2 = a\cos t + b\sin t \tag{27.123}$$

Differentiating three times,

$$y_2' = -a\sin t + b\cos t \tag{27.124}$$

$$y_2'' = -a\cos t - b\sin t \tag{27.125}$$

$$y_2''' = a\sin t - b\cos t \tag{27.126}$$

Substituting into the differential equation for y_2

$$3\cos t = a\sin t - b\cos t - 4(-a\sin t + b\cos t) \tag{27.127}$$

$$= a\sin t - b\cos t + 4a\sin t - 4b\cos t \tag{27.128}$$

$$= 5a\sin t - 5b\cos t \tag{27.129}$$

Equating coefficients gives $a = 0$ and $b =$ -3/5, and hence

$$y_2 = -\frac{3}{5}\sin t \tag{27.130}$$

For y_3, Since $r = -2$ is already a root of the characteristic equation with multiplicity one, we will try

$$y_3 = ate^{-2t} \tag{27.131}$$

Differentiating three times gives

$$\begin{aligned}
y_3' &= a\left(e^{-2t} - 2te^{-2t}\right) && (27.132)\\
&= ae^{-2t}(1-2t) && (27.133)\\
y_3'' &= a\left[-2e^{-2t}(1-2t) + e^{-2t}(-2)\right] && (27.134)\\
&= ae^{-2t}(-4+4t) && (27.135)\\
y_3''' &= a\left[-2e^{-2t}(-4+4t) + e^{-2t}(4)\right] && (27.136)\\
&= ae^{-2t}(12-8t) && (27.137)
\end{aligned}$$

Substituting for y_3 into its ODE gives

$$e^{-2t} = \left[ae^{-2t}(12-8t)\right] - 4\left[ae^{-2t}(1-2t)\right] = 8ae^{-2t} \tag{27.138}$$

and therefore $a = 1/8$, so that

$$y_3 = \frac{1}{8}te^{-2t} \tag{27.139}$$

Combining all of the particular solutions with the homogeneous solution gives us the general solution to the differential equation, which is given by

$$y = C_1 + C_2e^{2t} + C_3e^{-2t} - \frac{1}{8}t^2 - \frac{3}{5}\sin t + \frac{1}{8}te^{-2t}. \quad \square \tag{27.140}$$

The Wronskian

In this section we generalize the definition of Wronskian to higher order equations. If $\{y_1, ..., y_k\}$ are any set of functions then we can form the matrix

$$W[y_1, ..., y_k](t) = \begin{pmatrix} y_1 & y_2 & \cdots & y_k \\ y_1' & y_2' & & y_k' \\ \vdots & & & \vdots \\ y_1^{(k-1)} & y_2^{(k-1)} & \cdots & y_k^{(k-1)} \end{pmatrix} \tag{27.141}$$

We use the square bracket notation $[\cdots]$ above to indicate that $\mathbf{W}$ depends on a set of functions enclosed by the bracket, and the usual parenthesis notation $(\cdots)$ to indicate that $\mathbf{W}$ depends on a single independent variable t. When it is clear from the context what we mean we will omit the $[\cdots]$ and write $\mathbf{W}(t)$ where we mean (implicitly) $\mathbf{W}[\cdots](t)$.

Denote the general nth order linear differential equation by

$$L_n y = a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_o(t)y = f(t) \tag{27.142}$$

and let $\{y_1, ..., y_k\}$ form a fundamental set of solutions to $L_n y = 0$. Recall that $y_1, \ldots, y_k$ form a fundamental set of solutions if they are linearly independent and every other solution can be written as a linear combination. If the functions in the $\mathbf{W}$ matrix are a fundamental set of solutions to a differential equation then (27.141) is called the **fundamental matrix** of $L_n y = 0$.

The determinant of (27.141), regardless of whether the solutions form a fundamental set, is called the **Wronskian**, which we will denote by $W(t)$.

$$W[y_1, ..., y_k](t) = \begin{vmatrix} y_1 & \cdots & y_k \\ \vdots & & \vdots \\ y_1^{(k-1)} & \cdots & y_k^{(k-1)} \end{vmatrix} = \det W \tag{27.143}$$

Again, we will omit the square brackets $[\cdots]$ and write the Wronskian as $W(t)$ when the set of functions it depends on is clear as $W(t)$. When the set of functions $\{y_1, ..., y_k\}$ form a fundamental set of solutions to $L_n y = 0$ we will call it **the Wronskian of the differential equation**.

If we calculate the Wronskian of a set of functions that is not linearly independent, then one of the functions can be expressed as a linear combination of all other functions, and consequently, one of the columns of the matrix will be a linear combination of all the other columns. When this happens,

the determinant will be zero. Thus the Wronskian of a linearly dependent set of functions will always be zero. In fact, as we show in the following theorems, the Wronskian will be nonzero if and only if the functions form a complete set of solutions to the same differential equation.

Example 27.8. Find the Wronskian of $y''' - 4y' = 0$.

The characteristic equation is $0 = r^3 - 4r = r(r^2 - 4) = r(r-2)(r+2)$ and a fundamental set of solutions are $y_1 = 1$, $y_2 = e^{2t}$, and $y_3 = e^{-2t}$. Their Wronskian is

$$W(t) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \tag{27.144}$$

$$= \begin{vmatrix} 1 & e^{2t} & e^{-2t} \\ 0 & 2e^{2t} & -2e^{-2t} \\ 0 & 4e^{2t} & 4e^{-2t} \end{vmatrix} \tag{27.145}$$

$$= \begin{vmatrix} 2e^{2t} & -2e^{-2t} \\ 4e^{2t} & 4e^{-2t} \end{vmatrix} = 16. \quad \square \tag{27.146}$$

Example 27.9. Find the Wronskian of $y''' - 8y'' + 16y' = 0$.

The characteristic equation is $0 = r^3 - 8r^2 + 16r = r(r^2 - 8r + 16) = r(r-4)^2$, so a fundamental set of solutions is $y_1 = 1$, $y_2 = e^{4t}$ and $y_3 = te^{4t}$. Therefore

$$W(t) = \begin{vmatrix} 1 & e^{4t} & te^{4t} \\ 0 & 4e^{4t} & e^{4t}(1+4t) \\ 0 & 16e^{4t} & 8e^{4t}(1+2t) \end{vmatrix} \tag{27.147}$$

$$= (4e^{4t})[8e^{4t}(1+2t)] - [e^{4t}(1+4t)](16e^{4t}) \tag{27.148}$$

$$= 16e^{8t} \tag{27.149}$$

Theorem 27.8. Suppose that $y_1, y_2, ..., y_n$ all satisfy the same higher order linear homogeneous differential equation $L_n y = 0$ on (a, b) Then $y_1, y_2, ..., y_n$ form a fundamental set of solutions if and only if for some $t_0 \in (a, b)$, $W[y_1, ..., y_n](t_0) \neq 0$.

Proof. Let $y_1, ..., y_n$ be solutions to $L_n y = 0$, and suppose that $W[y_1, ..., y_n](t_0) \neq 0$ for some number $t_0 \in (a, b)$.

We need to show that $y_1, ..., y_n$ form a fundamental set of solutions. This means proving that any solution to $L_n y = 0$ has the form

$$\phi(t) = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n \tag{27.150}$$

Consider the initial value problem

$$L_n y = 0, \ y(t_0) = y_0, ..., y^{(n-1)}(t_0) = y_n \tag{27.151}$$

Certainly every $\phi(t)$ given by (27.150) satisfies the differential equation; we need to show that for some set of constants $C_1, ..., C_n$ it also satisfies the initial conditions.

Differentiating (27.150) $n-1$ times and combining the result into a matrix equation,

$$\begin{pmatrix} \phi(t_0) \\ \phi'(t_0) \\ \vdots \\ \phi^{(n-1)}(t_0) \end{pmatrix} = \begin{pmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & & y_n'(t_0) \\ \vdots & & & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{pmatrix} \tag{27.152}$$

The matrix on the right hand side of equation (27.152) is $\mathbf{W}[y_1, ..., y_n](t_0)$. By assumption, the determinant $W[y_1, ..., y_n](t_0) \neq 0$, hence the corresponding matrix $\mathbf{W}[y_1, ..., y_n](t_0)$ is invertible. Since $\mathbf{W}[y_1, ..., y_n](t_0)$ is invertible, there is a solution $\{C_1, ..., C_n\}$ to the equation

$$\begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix} = \mathbf{W}[y_1, ..., y_n](t_0) \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} \tag{27.153}$$

given by

$$\begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} = \{\mathbf{W}[y_1, .., y_n](t_0)\}^{-1} \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix} \tag{27.154}$$

Hence there exists a non-trivial set of numbers $\{C_1, ..., C_n\}$ such that

$$\phi(t) = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n \tag{27.155}$$

satisfies the initial value problem (27.151).

By uniqueness, every solution of this initial value problem must be identical to (27.155), and this means that it must be a linear combination of the $\{y_1, ..., y_n\}$.

Thus every solution of the differential equation is also a linear combination of the $\{y_1, ..., y_n\}$, and hence $\{y_1, ..., y_n\}$ must form a fundamental set of solutions.

To prove the converse, suppose that $y_1, ..., y_n$ are a fundamental set of solutions. We need to show that for some number $t_0 \in (a, b)$, $W[y_1, ..., y_n](t_0) \neq$

0. Since $y_1, ..., y_n$ form a fundamental set of solutions, any solution to the initial value problem (27.151) must have the form

$$\phi(t) = C_1 y_1 + C_2 y_2 + \cdots + C_n y_n \tag{27.156}$$

for some set of constants $\{C_1, ..., C_n\}$. Hence there must exist constants $C_1, ..., C_n$ such that

$$\begin{aligned} C_1 y_1(t_0) + \cdots + C_n y_n(t_0) &= y_0 \\ C_1 y_1'(t_0) + \cdots + C_n y_n'(t_0) &= y_1 \\ &\vdots \\ C_1 y_1^{(n-1)}(t_0) + \cdots + C_n y_n^{(n-1)}(t_0) &= y_{n-1} \end{aligned} \tag{27.157}$$

i.e., there is a solution $\{C_1, ..., C_n\}$ to

$$\begin{pmatrix} y_1(t_0) & \cdots & y_n(t_0) \\ \vdots & & \vdots \\ y_1^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{pmatrix} \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} = \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix} \tag{27.158}$$

This is only true if the matrix

$$\mathbf{W}[y_1, ..., y_n](t_0) = \begin{pmatrix} y_1(t_0) & \cdots & y_n(t_0) \\ \vdots & & \vdots \\ y_1^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{pmatrix} \tag{27.159}$$

is invertible, which in turn is true if and only if its determinant is nonzero. But the determinant is the Wronskian, hence there exists a number t_0 such that the Wronskian $W[y_1, ..., y_n](t_0) \neq 0$. □

Theorem 27.9. Suppose $y_1, ..., y_n$ are solutions $L_n y = 0$ on an interval (a, b), and let their Wronskian be denoted by $W[y_1, ..., y_n](t)$. Then the following are equivalent:

1. $W[y_1, ..., y_n](t) \neq 0 \ \ \forall t \in (a, b)$
2. $\exists t_0 \in (a, b)$ such that $W[y_1, ..., y_n](t_0) \neq 0$
3. $y_1, ..., y_n$ are linearly independent functions on (a, b).
4. $y_1, ..., y_n$ are a fundamental set of solutions to $L_n y = 0$ on (a, b).

Example 27.10. Two solutions of the differential equation $2t^2 y'' + 3ty' - y = 0$ are $y_1 = t^{1/2}$ and $y_2 = 1/t$; this can be verified by substitution into

the differential equation. Over what domain do these two functions form a fundamental set of solutions?

Calculating the Wronskian, we find that

$$W(t) = \begin{vmatrix} t^{1/2} & 1/t \\ 1/(2t^{1/2}) & -1/t^2 \end{vmatrix} \quad (27.160)$$

$$= -\frac{t^{1/2}}{t^2} - \frac{1}{t(2t^{1/2})} \quad (27.161)$$

$$= \frac{-3}{2t^{3/2}} \quad (27.162)$$

This Wronskian is never equal to zero. Thus these two solutions form a fundamental set on any open interval over which they are defined, namely $t > 0$ or $t < 0$. □

Theorem 27.10 (Abel's Formula)**.** The Wronskian of

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y + a_0(t) = 0 \quad (27.163)$$

is

$$W(t) = Ce^{-\int p(t)dt} \quad (27.164)$$

where $p(t) = a_{n-1}(t)/a_n(t)$. In particular, for $n = 2$, the Wronskian of

$$y'' + p(t)y' + q(t)y = 0 \quad (27.165)$$

is also given by the same formula, $W(t) = Ce^{-\int p(t)dt}$.

Lemma 27.11. Let $\mathbf{M}$ be an $n \times n$ square matrix with row vectors m_i, determinant M, and let $d(M, i)$ be the same matrix with the ith row vector replaced by dm_i/dt. Then

$$\frac{d}{dt} \det M = \sum_{i=1}^{n} \det d(M, i) \quad (27.166)$$

Proof. For n=2,

$$\frac{dM}{dt} = \frac{d}{dt}(m_{11}m_{22} - m_{12}m_{21}) \quad (27.167)$$

$$= m_{11}m_{22}' + m_{11}'m_{22} - m_{12}m_{21}' - m_{12}'m_{21} \quad (27.168)$$

Now assume that (27.166) is true for any $n \times n$matrix, and let $\mathbf{M}$ be any $n+1 \times n+1$ matrix. Then if we expand its determinant by the first row,

$$M = \sum_{i=1}^{n+1} (-1)^{1+i} m_{1i} \min(m_{1i}) \quad (27.169)$$

where $\min(m_{ij})$ is the minor of the ijth element. Differentiating,

$$\frac{dM}{dt} = \sum_{i=1}^{n+1}(-1)^{1+i}m'_{1i}\min(m_{1i}) + \sum_{i=1}^{n+1}(-1)^{1+i}m_{1i}\frac{d}{dt}\min(m_{1i}) \quad (27.170)$$

The first sum is $d(\mathbf{M},1)$. Since (27.166) is true for any $n \times n$ matrix, we can apply it to $\min(m_{1i})$ in the second sum.

$$\frac{dM}{dt} = d(M,1) + \sum_{i=1}^{n+1}(-1)^{1+i}m_{1i}\sum_{j=1}^{n} d(\min(m_{1i}), j) \quad (27.171)$$

which completes the inductive proof of the lemma. □

Proof. (Abel's Formula)

($n = 2$). Suppose y_1 and y_2 are solutions of (27.165). Their Wronskian is

$$W(t) = y_1 y_2' - y_2 y_1' \quad (27.172)$$

Differentiating,

$$W'(x) = y_1 y_2'' + y_1' y_2' - y_2' y_1' - y_2 y_1'' = y_1 y_2'' - y_2 y_1'' \quad (27.173)$$

Since $L_2 y_1 = L_2 y_2 = 0$,

$$y_1'' = -p(t)y_1' - q(t)y_1 \quad (27.174)$$

$$y_2'' = -p(t)y_2' - q(t)y_2 \quad (27.175)$$

Hence

$$\begin{aligned} W'(t) &= y_1(-p(t)y_2' - q(t)y_2) - y_2(-p(t)y_1' - q(t)y_1) \\ &= -p(t)(y_1 y_2' - y_2 y_1') \\ &= -p(t)W(t) \end{aligned} \quad (27.176)$$

Rearranging and integrating gives $W(t) = C\exp\left[-\int p(t)dt\right]$.

General Case. The Wronskian is

$$W[y_1, ..., y_n](t) = \begin{vmatrix} y_1 & \cdots & y_n \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \quad (27.177)$$

By the lemma, to obtain the derivative of a determinant, we differentiate row by row and add the results, hence

$$\begin{aligned}\frac{dW}{dt} &= \begin{vmatrix} y_1' & \cdots & y_n' \\ y_1' & & y_n' \\ y_1'' & & y_n'' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} y_1' & \cdots & y_n' \\ y_1'' & & y_n'' \\ y_1'' & & y_n'' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \\ &\quad + \cdots + \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & & y_n' \\ y_1'' & & y_n'' \\ \vdots & & \vdots \\ y_1^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} \end{aligned} \tag{27.178}$$

Every determinant except for the last contains a repeated row, and since the determinant of a matrix with a repeated row is zero, the only nonzero term is the last term.

$$\frac{dW}{dt} = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & & y_n' \\ y_1'' & & y_n'' \\ \vdots & & \vdots \\ y_1^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} \tag{27.179}$$

Since each y_j is a solution of the homogeneous equation,

$$y_j^{(n)} = -\frac{a_{n-1}(t)}{a_n(t)} y_j^{(n-1)} - \frac{a_{n-2}(t)}{a_n(t)} y_j^{(n-2)} - \cdots - \frac{a_0(t)}{a_n(t)} y_j \tag{27.180}$$

$$= -\frac{1}{a_n(t)} \sum_{i=0}^{n-1} a_i(t) y_j^{(i)} \tag{27.181}$$

Hence

$$\frac{dW}{dx} = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & & y_n' \\ \vdots & & \vdots \\ y_1^{(n-2)} & & y_n^{(n-2)} \\ -\dfrac{1}{a_n(t)}\displaystyle\sum_{i=0}^{n-1} a_i(t)y_1^{(i)} & \cdots & -\dfrac{1}{a_n(t)}\displaystyle\sum_{i=0}^{n-1} a_i(t)y_m^{(i)} \end{vmatrix} \tag{27.182}$$

The value of a determinant is unchanged if we add a multiple of one to another. So multiply the first row by$a_0(t)/a_n(t)$, the second row by$a_2(t)/a_n(t)$, etc., and add them all to the last row to obtain

$$\frac{dW}{dt} = \begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & & y_n' \\ \vdots & & \vdots \\ y_1^{(n-2)} & & y_n^{(n-2)} \\ -p(t)y_1^{(n-1)} & \cdots & -p(t)y_n^{(n-1)} \end{vmatrix} \tag{27.183}$$

where $p(t) = a_{n-1}(t)/a_n(t)$. We can factor a constant out of every element of a single row of the determinant if we multiply the resulting (factored) determinant by the same constant.

$$\frac{dW}{dt} = -p(t)\begin{vmatrix} y_1 & \cdots & y_n \\ y_1' & & y_n' \\ \vdots & & \vdots \\ y_1^{(n-2)} & & y_n^{(n-2)} \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = -p(t)W(t) \tag{27.184}$$

Integrating this differential equation achieves the desired formula for W. □

Example 27.11. Calculate the Wronskian of $y'' - 9y = 0$.

Since $p(t) = 0$, Abel's formula gives $W = Ce^{-\int 0\cdot dt} = C$. □

Example 27.12. Use Abel's formula to compute the Wronskian of $y''' - 2y'' - y' - 3y = 0$

This equation has $p(t) = -2$, and therefore $W = Ce^{-\int(-2)dt} = Ce^{2t}$. □

Example 27.13. Compute the Wronskian of $x^2y^{(??)} + xy^{(??)} + y'' - 4x = 0$.

We have $p(t) = t/t^2 = 1/t$. Therefore $W = Ce^{-\int(1/t)dt} = Ce^{-\ln t} = C/t$. □

Example 27.14. Find the general solution of

$$ty''' - y'' - ty' + y = 0 \tag{27.185}$$

given that $y = e^t$ and $y = e^{-t}$ are solutions.
From Abel's formula,

$$W(t) = \exp\left\{-\int(-1/t)dt\right\} = \exp\{\ln t\} = t \tag{27.186}$$

By direct calculation,

$$W(t)=\begin{vmatrix} e^{-t} & e^t & y \\ -e^{-t} & e^t & y' \\ e^{-t} & e^t & y'' \end{vmatrix} \tag{27.187}$$

$$= e^{-t}\begin{vmatrix} e^t & y' \\ e^t & y'' \end{vmatrix} + e^{-t}\begin{vmatrix} e^t & y \\ e^t & y'' \end{vmatrix} + e^{-t}\begin{vmatrix} e^t & y \\ e^t & y' \end{vmatrix} \tag{27.188}$$

$$= e^{-t}[(e^t y'' - e^t y') + (e^t y'' - e^t y) + (e^t y' - e^t y)] \tag{27.189}$$

$$= y'' - y' + y'' - y + y' - y \tag{27.190}$$

$$= 2y'' - 2y \tag{27.191}$$

Setting the two expressions for the Wronskian equal to one another,

$$y'' - y = \frac{t}{2} \tag{27.192}$$

The method of undetermined coefficients tells us that

$$y = C_1 e^{-t} + C_2 e^t - \frac{1}{2}t \tag{27.193}$$

is a solution (the first two terms are y_H and the third is y_P). $\square$

Variation of Parameters

Theorem 27.12 (Variation of Parameters.)**.** Suppose that $y_1, ..., y_n$ are a fundamental set of solutions to

$$L_n y = a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y = 0 \tag{27.194}$$

Then a particular solution to

$$L_n y = a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y = f(t) \tag{27.195}$$

is given by

$$\begin{aligned} y_P = y_1 \int_t \frac{W_1(s)f(s)}{W(s)a_n(s)} ds + y_2 \int_t \frac{W_2(s)f(s)}{W(s)a_n(s)} ds \\ + \cdots + y_n \int_t \frac{W_n(s)f(s)}{W(s)a_n(s)} ds \end{aligned} \tag{27.196}$$

where $W_j(t)$ is the determinant of $\mathbf{W}[y_1, ..., y_n](t)$ with the jth column replaced by a vector with all zeroes except for a 1 in the last row. In particular, for $n = 2$, a particular solution to $a(t)y'' + b(t)y' + c(t)y = f(t)$ is

$$y_p = -y_1(t) \int_t \frac{y_2(s)f(s)}{W(s)a(s)} ds + y_2(t) \int_t \frac{y_1(s)f(s)}{W(s)a(s)} ds \tag{27.197}$$

Proof for general case. Look for a solution of the form

$$y = u_1 y_1 + \cdots + u_n y_n \tag{27.198}$$

This is under-determined so we can make additional assumptions; in particular, we are free to assume that

$$\begin{aligned} u_1' y_1 + \cdots + u_n' y_n &= 0 \\ u_1' y_1' + \cdots + u_n' y_n' &= 0 \\ &\vdots \\ u_1' y_1^{(n-2)} + \cdots + u_n' y_n^{(n-2)} &= 0 \end{aligned} \tag{27.199}$$

Then

$$\begin{aligned} y' &= u_1 y_1' + \cdots + u_n y_n' \\ y'' &= u_1 y_1'' + \cdots + u_n y_n'' \\ &\vdots \\ y^{(n-1)} &= u_1 y_1^{(n-1)} + \cdots + u_n y_n^{(n-1)} \\ y^{(n)} &= u_1 y_1^{(n)} + \cdots + u_n y_n^{(n)} + u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)} \end{aligned} \tag{27.200}$$

So that

$$\begin{aligned}
f(t) &= a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y \\
&= a_n(t)\left[u_1 y_1^{(n)} + \cdots + u_n y_n^{(n)} + u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)}\right] + \\
&\quad a_{n-1}(t)\left[u_1 y_1^{(n-1)} + \cdots + u_n y_n^{(n-1)}\right] + \cdots + \\
&\quad a_1(t)\left[u_1 y_1' + \cdots + u_n y_n'\right] + a_0(t)\left[u_1 y_1 + \cdots + u_n y_n\right] \\
&= a_n(t)\left[u_1' y_1^{(n-1)} + \cdots + u_n' y_n^{(n-1)}\right]
\end{aligned} \tag{27.201}$$

Combining (27.201) and (27.199) in matrix form,

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & & & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_{n-1}' \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t)/a_n(t) \end{pmatrix} \tag{27.202}$$

The matrix on the left is the fundamental matrix of the differential equation, and hence invertible, so that

$$\begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_{n-1}' \\ u_n' \end{pmatrix} = \begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & & y_n' \\ \vdots & & \vdots \\ y_1^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t)/a_n(t) \end{pmatrix} = \frac{f(t)}{a_n(t)} \begin{pmatrix} [W^{-1}]_{1n} \\ [W^{-1}]_{2n} \\ \vdots \\ [W^{-1}]_{nn} \end{pmatrix} \tag{27.203}$$

where $\mathbf{W}$ is the fundamental matrix and $[W^{-1}]_{ij}$ denotes the *ijth* element of $\mathbf{W}^{-1}$.

$$[W^{-1}]_{jn} = \frac{cof[W]_{ni}}{\det W} = \frac{W_i(t)}{W(t)} \tag{27.204}$$

Therefore

$$\frac{du_i}{dt} = \frac{f(t)W_i(t)}{a_n(t)W(t)} \tag{27.205}$$

$$u_i(t) = \int_t \frac{f(s)W_i(s)}{a_n(s)W(s)} ds \tag{27.206}$$

Substitution of equation (27.206) into equation (27.198) yields equation (27.196). $\square$

Example 27.15. Solve $y''' + y' = \tan t$ using variation of parameters.

The characteristic equation is $0 = r^3 + r = r(r+i)(r-i)$; hence a fundamental set of solutions are $y_1 = 1$, $y_2 = \cos t$, and $y_3 = \sin t$. From either Abel's formula or a direct calculation, $W(t) = 1$, since

$$W = \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = 1 \tag{27.207}$$

$$y_P = y_1 \int_t \frac{W_1(s)f(s)}{W(s)a_3(s)} ds + y_2 \int_t \frac{W_2(s)f(s)}{W(s)a_3(s)} ds + y_3 \int_t \frac{W_3(s)f(s)}{W(s)a_3(s)} ds \tag{27.208}$$

where $a_3(s) = 1$, $f(s) = \tan s$, and

$$W_1 = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix} = 1 \tag{27.209}$$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & 1 & -\sin t \end{vmatrix} = -\cos t \tag{27.210}$$

$$W_3 = \begin{vmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & 1 \end{vmatrix} = -\sin t \tag{27.211}$$

Therefore

$$y_P = \int_t \tan s ds - \cos t \int_t \cos s \tan s ds - \sin t \int_t \sin s \tan s ds \tag{27.212}$$

Integrating the first term and substituting for the tangent in the third term,

$$y_P = -\ln|\cos t| - \cos t \int_t \sin s ds - \sin t \int_t \frac{\sin^2 s}{\cos s} ds \tag{27.213}$$

The second integral can not be integrated immediately, and the final integral can be solved by substituting $\sin^2 s = 1 - \cos^2 s$

$$y_P = -\ln|\cos t| + \cos^2 t - \sin t \int_t \frac{1-\cos^2 s}{\cos s} ds \tag{27.214}$$

Since the first term (the constant) is a solution of the homogeneous equation, we can drop it from the particular solution, giving

$$y_P = \ln|\cos t| - \sin t \ln|\sec t + \tan t| \tag{27.215}$$

and a general solution of

$$y = \ln|\cos t| - \sin t \ln|\sec t + \tan t| + C_1 + C_2 \cos t + C_3 \sin t. \quad \square \tag{27.216}$$

Lesson 28

Series Solutions

In many cases all we can say about the solution of

$$\left.\begin{aligned} a(t)y'' + b(t)y' + c(t)y = f(t) \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \end{aligned}\right\} \tag{28.1}$$

is a statement about whether or not a solution exists. So far, however, we do not have any generally applicable technique to actually find the solution. If a solution does exist, we know it must be twice differentiable (n times differentiable for an *nth* order equation).

If the solution is not just twice, but infinitely, differentiable, we call it an **analytic** function. According to Taylor's theorem, any function that is analytic at a point $t = t_0$ can be expanded in a power series

$$y(t) = \sum_{k=0}^{\infty} a_k (t - t_0)^k \tag{28.2}$$

where

$$a_k = \frac{y^{(k)}(t_0)}{k!} \tag{28.3}$$

Because of Taylor's theorem, the term analytic is sometimes used to mean that a function can be expanded in a power series at a point (analytic $\iff$ infinitely differentiable $\iff$ power series exists).

The **method of series** for solving differential equations solutions looks for analytic solutions to (28.1) by substituting equation (28.2) into the differential equation and solving for the coefficients $a_0, a_1, \ldots$

If we can somehow find a non-trivial solution for the coefficients then we have solved the initial value problem. In practice, we find the coefficients by a generalization of the **method of undetermined coefficients**.

Example 28.1. Solve the separable equation

$$\left.\begin{aligned} ty' - (t+1)y = 0 \\ y(0) = 0 \end{aligned}\right\} \tag{28.4}$$

by expanding the solution as a power series about the point $t_0 = 0$ and show that you get the same solution as by the method of separation of variables.

Separating variables, we can rewrite the differential equation as $dy/y = [(t+1)/t]dt$; integrating yields

$$\ln|y| = t + \ln|t| + C \tag{28.5}$$

hence a solution is

$$y = cte^t \tag{28.6}$$

for any value of the constant c. We should obtain the same answer using the method of power series.

To use the method of series, we begin by letting

$$y(t) = \sum_{k=0}^{\infty} a_k t^k \tag{28.7}$$

be our proposed solution, for some unknown (to be determined) numbers $a_0, a_1, \dots$. Then

$$y' = \sum_{k=0}^{\infty} k a_k t^{k-1} \tag{28.8}$$

Substituting (28.7) and (28.8) into (28.4),

$$0 = ty' - (t+1)y = t\sum_{k=0}^{\infty} k a_k t^{k-1} - (t+1)\sum_{k=0}^{\infty} a_k t^k \tag{28.9}$$

Changing the index of the second sum only to $k = m - 1$,

$$0 = \sum_{k=0}^{\infty} k a_k t^k - \sum_{k=0}^{\infty} a_k t^{k+1} - \sum_{k=0}^{\infty} a_k t^k \tag{28.10}$$

$$= \sum_{k=0}^{\infty} k a_k t^k - \sum_{m=1}^{\infty} a_{m-1} t^m - \sum_{k=1}^{\infty} a_k t^k \tag{28.11}$$

$$= \sum_{k=1}^{\infty} (k-1) a_k t^k - \sum_{k=1}^{\infty} a_{k-1} t^k \tag{28.12}$$

$$= \sum_{k=1}^{\infty} [(k-1) a_k - a_{k-1}] t^k \tag{28.13}$$

In the last line we have combined the two sums into a single sum over k. Equation (28.13) must hold for all t. But the functions $\{1, t, t^2, ...\}$ are linearly independent, and there is no non-trivial set of constants $\{C_k\}$ such that $\sum_{k=0}^{\infty} C_k t^k = 0$, i.e., $C_k = 0$ for all k. Hence

$$a_0 = 0 \tag{28.14}$$

$$(k-1) a_k - a_{k-1} = 0, \quad k = 1, 2, ... \tag{28.15}$$

Equation (28.15) is a recursion relation for a_k; it is more convenient to write it as $(k-1)a_k = a_{k-1}$. For the first several values of k it gives:

$$k = 1 : 0 \cdot a_1 = a_0 \tag{28.16}$$

$$k = 2 : 1 \cdot a_2 = a_1 \Rightarrow a_2 = a_1 \tag{28.17}$$

$$k = 3 : 2 \cdot a_3 = a_2 \Rightarrow a_3 = \frac{1}{2} a_2 = \frac{1}{2} a_1 \tag{28.18}$$

$$k = 4 : 3 \cdot a_4 = a_3 \Rightarrow a_4 = \frac{1}{3} a_3 = \frac{1}{3 \cdot 2} a_1 \tag{28.19}$$

$$k = 5 : 4 \cdot a_5 = a_4 \Rightarrow a_5 = \frac{1}{4} a_4 = \frac{1}{4 \cdot 3 \cdot 2} a_1 \tag{28.20}$$

$$\vdots$$

The general form appears to be $a_k = a_1/(k-1)!$; this is easily proved by induction, since the recursion relationship gives

$$a_{k+1} = a_k/k = a_1/[k \cdot (k-1)!] = a_1/k! \tag{28.21}$$

by assuming $a_k = a_1/(k-1)!$ as an inductive hypothesis. Hence

$$y = a_1 t + a_2 t^2 + a_3 t^3 + \cdots \tag{28.22}$$

$$= a_1 \left(t + t^2 + \frac{1}{2} t^3 + \frac{1}{3!} t^4 + \frac{1}{4!} t^5 + \cdots \right) \tag{28.23}$$

$$= a_1 t \left(1 + t + \frac{1}{2} t^2 + \frac{1}{3!} t^3 + \frac{1}{4!} t^4 + \cdots \right) \tag{28.24}$$

$$= a_1 t e^t \tag{28.25}$$

which is the same solution we found by separation of variables. We have not applied the initial condition, but we note in passing the only possible initial condition that this solution satisfies at $t_0 = 0$ is $y(0) = 0$. ☐

Example 28.2. Find a solution to the initial value problem

$$\left. \begin{aligned} y'' - ty' - y &= 0 \\ y(0) &= 1 \\ y'(0) &= 1 \end{aligned} \right\} \tag{28.26}$$

Letting $y = \sum_{k=0}^{\infty} c_k t^k$ and differentiating twice we find that

$$y' = \sum_{k=0}^{\infty} k c_k t^{k-1} \tag{28.27}$$

$$y'' = \sum_{k=0}^{\infty} k(k-1) c_k t^{k-2} \tag{28.28}$$

The initial conditions tell us that

$$c_0 = 1, \tag{28.29}$$

$$c_1 = 1 \tag{28.30}$$

Substituting into the differential equation.

$$0 = \sum_{k=0}^{\infty} k(k-1) c_k t^{k-2} - t \sum_{k=0}^{\infty} k c_k t^{k-1} - \sum_{k=0}^{\infty} c_k t^k \tag{28.31}$$

Let $j = k-2$ in the first term, and combining the last two terms into a single sum,

$$0 = \sum_{j=-2}^{\infty} (j+2)(j+1) c_{j+2} t^j - \sum_{k=0}^{\infty} (k+1) c_k t^k \tag{28.32}$$

Since the first two terms (corresponding to $j = -2$ and $j = -1$) in the first sum are zero, we can start the index at $j = 0$ rather than $j = -2$. Renaming the index back to k, and combining the two series into a single sum,

$$0=\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}t^k - \sum_{k=0}^{\infty}(k+1)c_k t^k \tag{28.33}$$

By linear independence,

$$(k+2)(k+1)c_{k+2} = (k+1)c_k \tag{28.34}$$

Rearranging, $c_{k+2} = c_k/(k+2)$; letting $j = k+2$ and including the initial conditions (equation (28.29)), the general recursion relationship is

$$c_0 = 1,\ c_1 = 1,\ \ c_k = c_{k-2}/k \tag{28.35}$$

Therefore

$$c_2 = \frac{c_0}{2} = \frac{1}{2} \qquad c_3 = \frac{c_1}{3} = \frac{1}{3} \tag{28.36}$$

$$c_4 = \frac{c_2}{4} = \frac{1}{4\cdot 2} \qquad c_5 = \frac{c_3}{5} = \frac{1}{5\cdot 3} \tag{28.37}$$

$$c_6 = \frac{c_4}{6} = \frac{1}{6\cdot 4\cdot 2} \qquad c_7 = \frac{c_5}{7} = \frac{1}{7\cdot 5\cdot 3} \tag{28.38}$$

$$\vdots$$

So the solution of the initial value problem is

$$y=c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots \tag{28.39}$$

$$=1 + \frac{1}{2}t^2 + \frac{1}{4\cdot 2}t^4 + \frac{1}{6\cdot 4\cdot 2}t^6 + \cdots \tag{28.40}$$

$$+t + \frac{1}{3}t^3 + \frac{1}{5\cdot 3}t^5 + \frac{1}{7\cdot 5\cdot 3}t^7 + \cdots \quad \square \tag{28.41}$$

Example 28.3. Solve the Legendre equation of order 2, given by

$$\left.\begin{aligned}(1-t^2)y'' - 2ty' + 6y = 0\\ y(0) = 1\\ y'(0) = 0\end{aligned}\right\} \tag{28.42}$$

We note that "order 2" in the name of equation (28.42) is not related to the order of differential equation, but to the fact that the equation is a member of a family of equations of the form

$$(1-t^2)y'' - 2ty' + n(n+1)y = 0 \tag{28.43}$$

where n is an integer ($n = 2$in this case). Expanding as usual we obtain

$$y = \sum_{k=0}^{\infty} c_k t^k \tag{28.44}$$

$$y' = \sum_{k=0}^{\infty} kc_k t^{k-1} \tag{28.45}$$

$$y'' = \sum_{k=0}^{\infty} k(k-1)c_k t^{k-2} \tag{28.46}$$

Substituting into the differential equation,

$$0 = (1-t^2)\sum_{k=0}^{\infty} c_k k(k-1)t^{k-2} - 2t\sum_{k=0}^{\infty} c_k k t^{k-1} + 6\sum_{k=0}^{\infty} c_k t^k \tag{28.47}$$

$$= \sum_{k=0}^{\infty} c_k k(k-1)t^{k-2} - \sum_{k=0}^{\infty} c_k k(k-1)t^k - 2t\sum_{k=0}^{\infty} c_k k t^{k-1} + 6\sum_{k=0}^{\infty} c_k k t^k \tag{28.48}$$

$$= \sum_{k=0}^{\infty} c_k k(k-1)t^{k-2} - 2t\sum_{k=0}^{\infty} c_k k t^{k-1} + \sum_{k=0}^{\infty}(6-k^2+k)c_k t^k \tag{28.49}$$

The first term can be rewritten as

$$\sum_{k=0}^{\infty} c_k k(k-1)t^{k-2} = \sum_{k=2}^{\infty} c_k k(k-1)t^{k-2} \tag{28.50}$$

$$= \sum_{m=0}^{\infty} c_{m+2}(m+1)(m+2)t^m \tag{28.51}$$

$$= \sum_{k=0}^{\infty} c_{k+2}(k+1)(k+2)t^k \tag{28.52}$$

and the middle term in (28.49) can be written as

$$-2t\sum_{k=0}^{\infty} c_k k t^{k-1} = \sum_{k=0}^{\infty}(-2)kc_k t^k \tag{28.53}$$

Hence

$$0 = \sum_{k=0}^{\infty} c_{k+2}(k+1)(k+2)t^k + \sum_{k=0}^{\infty}(-2)kc_k t^k + \sum_{k=0}^{\infty}(6-k^2+k)c_k t^k \tag{28.54}$$

$$= \sum_{k=0}^{\infty} c_{k+2}(k+1)(k+2)t^k + \sum_{k=0}^{\infty}(k^2+k-6)c_k t^k \tag{28.55}$$

$$= \sum_{k=1}^{\infty} \left[c_{k+2}(k+1)(k+2) - (k+3)(k-2)c_k\right] t^k \tag{28.56}$$

By linear independence

$$c_{k+2} = \frac{(k+3)(k-2)}{(k+2)(k+1)} c_k,\ k = 0, 1, ... \tag{28.57}$$

or

$$c_k = \frac{(k+1)(k-4)}{k(k-1)} c_{k-2},\ k = 2, 3, ... \tag{28.58}$$

From the initial conditions we know that $c_0 = 1$ and $c_1 = 0$. Since c_3, c_5, ... are all proportional to c_1, we conclude that the odd-indexed coefficients are all zero.

Starting with $k = 2$ the even-indexed coefficients are

$$\left.\begin{aligned} c_2 = \frac{(2+1)(2-4)}{2(2-1)} c_0 = -3c_0 = -3 \\ c_4 = \frac{(4+1)(4-4)}{4(4-1)} c_2 = 0 \\ c_6 = c_8 = c_{10} = \cdots = 0 \end{aligned}\right\} \tag{28.59}$$

Hence

$$y = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots = 1 - 3t^2 \tag{28.60}$$

which demonstrates that sometimes the infinite series terminates after a finite number of terms. □

Ordinary and Singular Points

The method of power series solutions we have described will work at any ordinary point of a differential equation. A point $t = t_0$ is called **an ordinary point** of the differential operator

$$L_n y = a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_0(t)y \tag{28.61}$$

if all of the functions

$$p_k(t) = \frac{a_k(t)}{a_n(t)} \tag{28.62}$$

are analytic at $t = t_0$, and is called a **singular point** (or **singularity**) of the differential operator if any of the $p_k(t)$ are not analytic at $t = t_0$. If not all the $p_k(t)$ are analytic at $t = t_0$ but all of the functions

$$q_k(t) = (t - t_0)^{n-k} p_k(t) \tag{28.63}$$

namely,

$$\left.\begin{aligned} q_{n-1}(t) &= (t - t_0)\frac{a_{n-1}(t)}{a_n(t)} \\ q_{n-2}(t) &= (t - t_0)^2\frac{a_{n-2}(t)}{a_n(t)} \\ &\vdots \\ q_0(t) &= (t - t_0)^n\frac{a_0(t)}{a_n(t)} \end{aligned}\right\} \tag{28.64}$$

are analytic, then the point is called a **regular** (or **removable**) **singularity**. If none of the $q_k(t)$ are analytic, then the point is called an **irregular singularity**. We will need to modify the method at regular singularities, and it may not work at all at irregular singularities.

For second order equations, we say the a point of $t = t_0$ is an **ordinary point** of

$$y'' + p(t)y' + q(t)y = 0 \tag{28.65}$$

if both $p(t)$ and $q(t)$ are analytic at $t = t_0$; a **regular singularity** if $(t - t_0)p(t)$ and $(t - t_0)^2 q(t)$ are analytic at $t = t_0$; and an **irregular singularity** if at least one of them is not analytic at $t = t_0$

Theorem 28.1. [Existence of a power series solution at an ordinary point] If $\{p_j(t)\}$, $j = 0, 1, ..., n-1$ are analytic functions at $t = t_0$ then the initial value problem

$$\left.\begin{aligned} y^{(n)} + p_{n-1}(t)y^{(n-1)} + p_{n-2}(t)y^{(n-2)} + \cdots + p_0(t)y = 0 \\ y(t_0) = y_0, y'(t_0) = y_1, ..., y^{n-1}(t_0) = y_n \end{aligned}\right\} \tag{28.66}$$

has an analytic solution at $t = t_0$, given by

$$y = \sum_{k=0}^{\infty} c_k (t - t_0)^k \tag{28.67}$$

where

$$c_k = \frac{y_k}{k!}, \quad k = 0, 1, ..., n-1 \tag{28.68}$$

and the remaining c_k may be found by the method of undetermined coefficients. Specifically, if $p(t)$ and $q(t)$ are analytic functions at $t = t_0$, then the second order initial value problem

$$\left.\begin{aligned} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_1 \end{aligned}\right\} \tag{28.69}$$

has an analytic solution

$$y = y_0 + y_1(t - t_0) + \sum_{k=2}^{\infty} c_k (t - t_0)^k \tag{28.70}$$

Proof. (for $n = 2$). Without loss of generality we will assume that $t_0 = 0$. We want to show that a non-trivial set of $\{c_j\}$ exists such that

$$y = \sum_{j=0}^{\infty} c_j t^j \tag{28.71}$$

To do this we will determine the conditions under which (28.71) is a solution of (28.69). If this series converges, then we can differentiate term by term,

$$y' = \sum_{j=0}^{\infty} j c_j t^{j-1} \tag{28.72}$$

$$y'' = \sum_{j=0}^{\infty} j(j-1) c_j t^{j-2} \tag{28.73}$$

Observing that the first term of the y' series and the first two terms of the y'' series are zero, we find, after substituting $k = j - 1$ in the first and $k = j - 2$ in the second series, that

$$y' = \sum_{k=0}^{\infty} (k+1) c_{k+1} t^k \tag{28.74}$$

$$y'' = \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} t^k \tag{28.75}$$

Since $p(t)$ and $q(t)$ are analytic then they also have power series expansions which we will assume are given by

$$p(t) = \sum_{j=0}^{\infty} p_j t^j \tag{28.76}$$

$$q(t) = \sum_{j=0}^{\infty} q_j t^j \tag{28.77}$$

with some radius of convergence R. Substituting (28.71), (28.74) and (28.76) into (28.69)

$$\begin{aligned} 0 = &\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}t^k + \left(\sum_{j=0}^{\infty} p_j t^j\right)\left(\sum_{k=0}^{\infty}(k+1)c_{k+1}t^k\right) \\ &+ \left(\sum_{j=0}^{\infty} q_j t^j\right)\left(\sum_{k=0}^{\infty} c_k t^k\right) \end{aligned} \tag{28.78}$$

Since the power series converge we can multiply them out term by term:

$$\begin{aligned} 0 = &\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}t^k + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} p_j t^j (k+1)c_{k+1}t^k \\ &+ \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} q_j t^j c_k t^k \end{aligned} \tag{28.79}$$

We next use the two identities,

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(k+1)c_{k+1}p_j t^{j+k} = \sum_{k=0}^{\infty}\sum_{j=0}^{k}(j+1)c_{j+1}p_{k-j}t^k \tag{28.80}$$

and

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} q_j c_k t^{k+j} = \sum_{k=0}^{\infty}\sum_{j=0}^{k} q_{k-j}c_j t^k \tag{28.81}$$

Substituting (28.80) and (28.81) into the previous result

$$0 = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}t^k \tag{28.82}$$

Since the $t^k$are linearly independent,

$$(k+2)(k+1)c_{k+2} = -\sum_{j=0}^{k}\left[(j+1)c_{j+1}p_{k-j} + c_j q_{k-j}\right] \tag{28.83}$$

By the triangle inequality,

$$|(k+2)(k+1)c_{k+2}| \le \sum_{j=0}^{k} \left[(j+1)\left|c_{j+1}\right|\left|p_{k-j}\right| + \left|c_j\right|\left|q_{k-j}\right|\right] \tag{28.84}$$

Choose any r such that $0 < r < R$, where R is the radius of convergence of (28.76). Then since the two series for p and q converge there is some number M such that

$$\left|p_j\right| r^j \le M, \quad \left|q_j\right| r^j \le M \tag{28.85}$$

Otherwise there would be a point within the radius of convergence at which the two series would diverge. Hence

$$|(k+2)(k+1)c_{k+2}| \le \frac{M}{r^k} \textstyle\sum_{j=0}^{k} r^j \left[(j+1)\left|c_{j+1}\right| + \left|c_j\right|\right] \tag{28.86}$$

where in the second line we are merely adding a positive number to the right hand side of the inequality. Define $C_0 = |c_0|$, $C_1 = |c_1|$, and define $C_3, C_4, \ldots$ by

$$(k+2)(k+1)C_{k+2} = \frac{M}{r^k} \sum_{j=0}^{k} r^j \left[(j+1)C_{j+1} + C_j\right] + MC_{k+1}r \tag{28.87}$$

Then since $|(k+2)(k+1)c_{k+2}| \le (k+2)(k+1)C_{k+2}$ we know that $|c_k| \le C_k$ for all k. Thus if the series $\sum_{k=0}^{\infty} C_k t^k$ converges then the series $\sum_{k=0}^{\infty} c_k t^k$ must also converge by the comparison test. We will do this by the ratio test. From (28.87)

$$k(k+1)C_{k+1} = \frac{M}{r^{k-1}} \sum_{j=0}^{k-1} r^j \left[(j+1)C_{j+1} + C_j\right] + MC_k r \tag{28.88}$$

$$k(k-1)C_k = \frac{M}{r^{k-2}} \sum_{j=0}^{k-2} r^j \left[(j+1)C_{j+1} + C_j\right] + MC_{k-1} r \tag{28.89}$$

Multiplying (28.88) by r, and writing the last term of the sum explicitly,

$$\begin{aligned} k(k+1)C_{k+1}r &= \frac{M}{r^{k-2}} \textstyle\sum_{j=0}^{k-1} r^j \left[(j+1)C_{j+1} + C_j\right] + MC_k r^2 \\ &= \frac{M}{r^{k-2}} \textstyle\sum_{j=0}^{k-2} r^j \left[(j+1)C_{j+1} + C_j\right] + Mr\left[kC_k + C_{k-1}\right] + MC_k r^2 \end{aligned} \tag{28.90}$$

Substituting equation (28.89) into equation (28.90)

$$rk(k+1)C_{k+1} \quad = \quad k(k-1)C_k - MC_{k-1}r + M\left[kC_k + C_{k-1}\right]r + MC_kr^2 \tag{28.91}$$

Dividing by $rk(k+1)C_k$ gives

$$\frac{C_{k+1}}{C_k} = \frac{k(k-1) + Mkr + Mr^2}{rk(k+1)} \tag{28.92}$$

Multiplying by $t = t^{k+1}/t^k$ and taking the limit as $k \to \infty$

$$\lim_{k\to\infty}\left|\frac{C_{k+1}t^{k+1}}{C_kt^k}\right| = \lim_{k\to\infty}\left|\frac{k(k-1) + Mkr + Mr^2}{rk(k+1)}t\right| \tag{28.93}$$

Dividing the numerator an denominator by k^2,

$$\lim_{k\to\infty}\left|\frac{C_{k+1}t^{k+1}}{C_kt^k}\right| = |t|\lim_{k\to\infty}\left|\frac{1 - 1/k + Mr/k + Mr^2/k^2}{r + r/k}\right| = \frac{|t|}{|r|} \tag{28.94}$$

Therefore by the ratio test $\sum_{k=0}^{\infty} C_kt^k$ converges for $|t| < r$, and hence by the comparison test $\sum_{k=0}^{\infty} c_kt^k$ also converges. Therefore there is an analytic solution to any second order homogeneous ordinary differential equation with analytic coefficients. The coefficients of the power series are given by $c_0 = y_0$, $c_1 = y_1$ (by Taylor's theorem) and the recursion relationship (28.83) for $c_2, c_3, \ldots$ □

Many of the best-studied differential equations of mathematical physics (see table 28.1) are most easily solved using the method of power series solutions, or the related method of Frobenius that we will discuss in the next section. A full study of these equations is well beyond the scope of these notes; many of their most important and useful properties derive from the fact that many of them arise from their solutions as boundary value problems rather than initial value problems. Our interest in these functions is only that they illustrate the method of power series solutions.

Example 28.4. Find the general solutions to **Airy's Equation**:

$$y'' = ty \tag{28.95}$$

Using our standard substitutions,

$$y = \sum_{k=0}^{\infty} a_kt^k \tag{28.96}$$

$$y'' = \sum_{k=0}^{\infty} k(k-1)t^{k-2} \tag{28.97}$$

Table 28.1: Table of Special Functions defined by Differential Equations.

Name of Equation	Differential Equation Names of Solutions
Airy Equation	$y'' = ty$ Airy Functions $Ai(t), Bi(t)$
Bessel Equation	$t^2y'' + ty' + (t^2 - \nu^2)y = 0,\ 2\nu \in \mathbb{Z}^+$ Bessel Functions $J_\nu(t)$ Neumann Functions $Y_\nu(t)$
Modified Bessel Equation	$t^2y'' + ty' - (t^2 + \nu^2)y = 0,\ \ \nu \in \mathbb{Z}^+$ Modified Bessel Functions $I_\nu(t)$, $K_\nu(x)$ Hankel Functions $H_\nu(t)$
Euler Equation	$t^2y'' + \alpha ty' + \beta y = 0,\ \alpha, \beta \in \mathbb{C}$ t^r, where $r(r-1) + \alpha r + \beta = 0$
Hermite Equation	$y'' - 2ty' + 2ny = 0,\ n \in \mathbb{Z}^+$ Hermite polynomials $H_n(t)$
Hypergeometric Equation	$t(1-t)y'' + (c - (a+b+1)t)y' - aby = 0, a, b, c, d \in \mathbb{R}$ Hypergeometric Functions F, ${}_2F_1$
Jacobi Equation	$t(1-t)y'' + [q - (p+1)t]y' + n(p+n)y = 0, n \in \mathbb{Z}, a, b \in \mathbb{R}$ Jacobi Polynomicals J_n
Kummer Equation	$ty'' + (b-t)y' - ay = 0, a, b \in \mathbb{R}$ Confluent Hypergeometric Functions ${}_1F_1$
Laguerre Equation	$ty'' + (1-t)y' + my = 0, m \in \mathbb{Z}^+$ Laguerre Polynomials $L_m(t)$
Associated Laguerre Eqn.	$ty'' + (k+1-t)y$ Associated Laguerre Polynomials $L_m^k(t)$
Legendre Equation	$(1-t^2)y'' - 2ty' + n(n+1)y = 0,\ n \in \mathbb{Z}^+$ Legendre Polynomials $P_n(t)$
Associated Legendre Eq.	$(1-t^2)y'' - 2ty' + \left[n(n+1) - \frac{m^2}{1-t^2}\right] y = 0, m, n \in \mathbb{Z}^+$ Associate Legendre Polynomials $P_n^m(t)$
Tchebysheff Equation	$(1-t^2)y'' - ty' + n^2y = 0$ (Type I) $(1-t^2)y'' - 3ty' + n(n+2)y = 0$ (Type II) Tchebysheff Polynomials $T_n(t), U_n(t)$

in the differential equation gives

$$\sum_{k=0}^{\infty} k(k-1)a_k t^{k-2} = t\sum_{k=0}^{\infty} a_k t^k = \sum_{k=0}^{\infty} a_k t^{k+1} \tag{28.98}$$

Renumbering the index to $j = k - 2$ (on the left) and $j = k + 1$ (on the right), and recognizing that the first two terms of the sum on left are zero, gives

$$\sum_{j=0}^{\infty}(j+2)(j+1)a_{j+2}t^j = \sum_{j=1}^{\infty} a_{j-1}t^j \tag{28.99}$$

Rearranging,

$$2a_2 + \sum_{j=1}^{\infty}\left[(j+2)(j+1)a_{j+2} - a_{j-1}\right]t^j = 0 \tag{28.100}$$

By linear independence, $a_2 = 0$ and the remaining a_j satisfy

$$(j+2)(j+1)a_{j+2} = a_{j-1} \tag{28.101}$$

If we let $k = j + 2$ and solve for a_k,

$$a_k = \frac{a_{k-3}}{k(k-1)} \tag{28.102}$$

The first two coefficients, a_0 and a_1, are determined by the initial conditions; all other coefficients follow from the recursion relationship.. In particular, since $a_2 = 0$, every third successive coefficient $a_2 = a_5 = a_8 = \cdots = 0$. Starting with a_0 and a_1 we can begin to tabulate the remaining ones:

$$a_3 = \frac{a_0}{3\cdot 2} \tag{28.103}$$

$$a_6 = \frac{a_3}{6\cdot 5} = \frac{a_0}{6\cdot 5\cdot 3\cdot 2} \tag{28.104}$$

$$a_9 = \frac{a_6}{9\cdot 8} = \frac{a_0}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2} \tag{28.105}$$

$$\vdots$$

and

$$a_4 = \frac{a_1}{4\cdot 3} \tag{28.106}$$

$$a_7 = \frac{a_4}{7\cdot 6} = \frac{a_1}{7\cdot 6\cdot 4\cdot 3} \tag{28.107}$$

$$a_{10} = \frac{a_7}{10\cdot 9} = \frac{a_1}{10\cdot 9\cdot 7\cdot 6\cdot 4\cdot 3} \tag{28.108}$$

$$\vdots$$

Thus

$$y = a_0 \left(1 + \frac{1}{6}t^3 + \frac{1}{180}t^6 + \frac{1}{12960}t^9 + \cdots\right) \quad (28.109)$$

$$+ a_1 t \left(1 + \frac{1}{12}t^3 + \frac{1}{504}t^6 + \frac{1}{45360}t^9 + \cdots\right) \quad (28.110)$$

$$= a_0 y_1(t) + a_1 y_2(t) \quad (28.111)$$

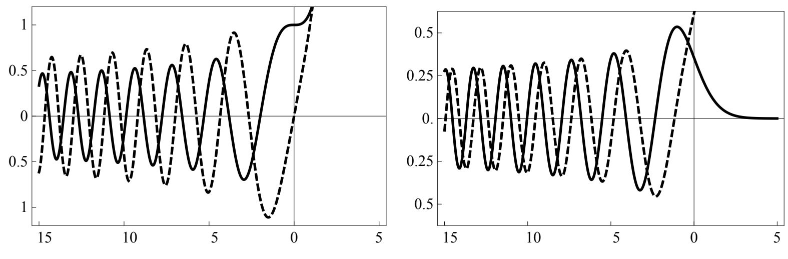
where y_1 and y_2 are defined by the sums in parenthesis. It is common to define the **Airy Functions**

$$\mathrm{Ai}(t) = \frac{1}{3^{2/3}\Gamma(2/3)} y_1(t) - \frac{1}{3^{1/3}\Gamma(1/3)} y_2(t) \quad (28.112)$$

$$\mathrm{Bi}(t) = \frac{\sqrt{3}}{3^{2/3}\Gamma(2/3)} y_1(t) + \frac{\sqrt{3}}{3^{1/3}\Gamma(1/3)} y_2(t) \quad (28.113)$$

Either the sets $\{y_1, y_2\}$ or $\{\mathrm{Ai}, \mathrm{Bi}\}$ are fundamental sets of solutions to the Airy equation. □

Figure 28.1: Solutions to Airy's equation. Left: The fundamental set y_1 (solid) and y_2(*dashed*). Right: the traditional functions $\mathrm{Ai}(t)$(solid) and $\mathrm{Bi}(t)$ (dashed). The renormalization keeps $\mathrm{Ai}(t)$ bounded, whereas the unnormalized solutions are both unbounded.

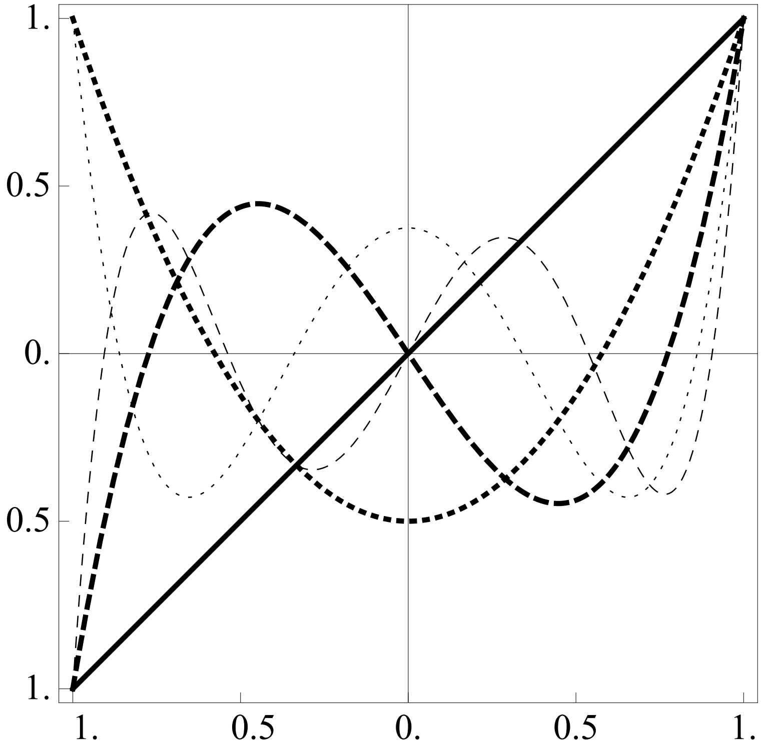


Example 28.5. Legendre's Equation of order n is given by

$$(1 - t^2)y'' - 2ty' + n(n+1)y = 0 \quad (28.114)$$

where n is any integer. Equation (28.114) is actually a family of differential equations for different values of n; we have already solved it for $n = 2$ in example 28.3, where we found that most of the coefficients in the power series solutions were zero, leaving us with a simple quadratic solution.

Figure 28.2: Several Legendre Polynomials. P_1 (Bold); P_2 (Bold, Dotted); P_3 (Bold, Dashed); P_4 (Thin, Dotted); P_5 (Thin, Dashed).



In general, the solutions of the nth equation (28.114) will give a polynomial of order n, called the **Legendre polynomial** $P_n(t)$.

To solve equation (28.114) we substitute

$$y = \sum_{k=0}^{\infty} a_k t^k \tag{28.115}$$

$$y' = \sum_{k=0}^{\infty} k a_k t^{k-1} \tag{28.116}$$

$$y'' = \sum_{k=0}^{\infty} k(k-1) a_k t^{k-2} \tag{28.117}$$

into (28.114) and collect terms,

$$0 = (1-t^2)\sum_{k=0}^{\infty} k(k-1)a_k t^{k-2} - 2t\sum_{k=0}^{\infty} k a_k t^{k-1} + n(n+1)\sum_{k=0}^{\infty} a_k t^k \tag{28.118}$$

Let $j = k-2$ in the first sum and observe that the first two terms in that sum are zero.

$$0 = \sum_{j=0}^{\infty}(j+1)(j+2)a_{j+2}t^j + \sum_{k=0}^{\infty}\left[-k-k^2+n(n+1)\right] a_k t^k \tag{28.119}$$

We can combine this into a single sum in t^k; by linear independence all of the coefficients must be zero,

$$(k+1)(k+2)a_{k+2} - [k(k+1)-n(n+1)]a_k = 0 \tag{28.120}$$

for all $k = 0, 1, 2, ...$, and therefore

$$a_{k+2} = \frac{k(k+1)-n(n+1)}{(k+1)(k+2)}a_k, k = 0, 1, 2, ... \tag{28.121}$$

The first two coefficients, a_0 and a_1 are arbitrary. The remaining ones are determined by (28.121), and generates two sequences of coefficients

$$\begin{array}{c} a_1, a_3, a_5, a_7, ... \\ a_0, a_2, a_4, a_6, ... \end{array} \tag{28.122}$$

so that we can write

$$y = \sum_{k\ even} a_k t^k + \sum_{k\ odd} a_k t^k \tag{28.123}$$

These two series are linearly independent. In particular, the right hand side of (28.121) vanishes when

$$k(k+1) = n(n+1) \tag{28.124}$$

so that for $k = n$ one of these two series will be a finite polynomial of order n. Normalized versions of the solutions are called the Legendre Polynomials, and the first few are given in the (28.125) through (28.135).

$$P_0(t) = 1 \tag{28.125}$$

$$P_1(t) = t \tag{28.126}$$

$$P_2(t) = \frac{1}{2}\left(3t^2 - 1\right) \tag{28.127}$$

$$P_3(t) = \frac{1}{2}\left(5t^3 - 3t\right) \tag{28.128}$$

$$P_4(t) = \frac{1}{8}\left(35t^4 - 30t^2 + 3\right) \tag{28.129}$$

$$P_5(t) = \frac{1}{8}\left(63t^5 - 70t^3 + 15t\right) \tag{28.130}$$

$$P_6(t) = \frac{1}{16}\left(231t^6 - 315t^4 + 105t^2 - 5\right) \tag{28.131}$$

$$P_7(t) = \frac{1}{16}\left(429t^7 - 693t^5 + 315t^3 - 35t\right) \tag{28.132}$$

$$P_8(t) = \frac{1}{128}\left(6435t^8 - 12012t^6 + 6930t^4 - 1260t^2 + 35\right) \tag{28.133}$$

$$P_9(t) = \frac{1}{128}\left(12155t^9 - 25740t^7 + 18018t^5 - 4620t^3 + 315t\right) \tag{28.134}$$

$$P_{10}(t) = \frac{1}{256}\left(46189t^{10} - 109395t^8 + 90090t^6 - 30030t^4 + 3465t^2 - 63\right) \tag{28.135}$$

Summary of Power series method.

To solve the linear differential equation

$$L_n y = a_n(t)y^{(n)} + \cdots + a_0(t)y = f(t) \tag{28.136}$$

or the corresponding initial value problem with

$$y(t_0) = y_0, ..., y^{(n-1)}(t_0) = y_n \tag{28.137}$$

as a power series about the point $t = t_0$

1. Let $y = \sum_{k=0}^{\infty} c_k(t - t_0)^k$
2. If initial conditions are given, use Taylor's theorem to assign the first n values of c_k as

$$c_k = y^{(k)}(t_0)/k! = y_k/k!,\ k = 0, 1, ..., n - 1 \tag{28.138}$$

3. Calculate the first n derivatives of y.
4. Substitute the expressions for $y, y', ..., y^{(n)}$ into (28.136).
5. Expand all of the $a_n(t)$that are not polynomials in Taylor series about $t = t_0$ and substitute these expansions into the expression obtained in step 4.
6. Multiply out any products of power series into a single power series.
7. By an appropriate renumbering of indices, combine all terms in the equation into an equation of the form $\sum_k u_k(t - t_0)^k = 0$ where each u_k is a function of some set of the c_k.
8. Use linear independence to set the $u_k = 0$ and find a relationship between the c_k.
9. The radius of convergence of power series is $\min_{\{t_i\}}(|t_0 - t_i|)$ where $\{t_i\}$ is the set of all singularities of $a(t)$.

Lesson 29

Regular Singularities

The method of power series solutions discussed in chapter 28 fails when the series is expanded about a singularity. In the special case of **regular singularity** this problem can be rectified with method of Frobenius, which we will discuss in chapter 30. We recall the definition of a regular singular point here.

Definition 29.1. Let

$$a(t)y'' + b(t)y' + c(t)y = 0 \tag{29.1}$$

have a singular point at t_0, i.e., $a(t_0) = 0$. Then if the limits

$$\lim_{t \to t_0} \frac{(t - t_0)b(t)}{a(t)} = L_1 \tag{29.2}$$

$$\lim_{t \to t_0} \frac{(t - t_0)^2 c(t)}{a(t)} = L_2 \tag{29.3}$$

exist and are finite we say that t_0 is a **regular singularity of the differential equation**. Note that these limits will always exist when the arguments of the limit are analytic (infinitely differentiable) at t_0. If either of the limits does not exist or are infinite (or the arguments of either limit is not analytic) then we say that t_0 is an **irregular singularity of the differential equation**. If $a(t_0) \neq 0$ then we call t_0 an **ordinary point of the differential equation**.

Theorem 29.2. The power series method of chapter 28 works whenever the series is expanded about an ordinary point, and the radius of convergence of the series is the distance from t_0 to the nearest singularity of $a(t)$.

Proof. This is a restatement of theorem 28.1. $\square$

Example 29.1. The differential equation $(1+t^2)y'' + 2ty' + 4t^2y = 0$ has singularities at $t = \pm i$. The series solution $\sum a_k t^k$ about $t_0 = 0$ has a radius of convergence of 1 while the series solution $\sum b_k(t-1)^k$ about $t_0 = 1$ has a radius of convergence of $\sqrt{2}$. $\square$

The **Cauchy-Euler equation**

$$t^2y'' + \alpha ty' + \beta y = 0 \tag{29.4}$$

where $\alpha, \beta \in \mathbb{R}$ is the canonical (standard or typical) example of a differential equation with a regular singularity at the origin. It is useful because it provides us with an insight into why the Frobenius method that we will discuss in chapter 30 will work, and the forms and methods of solution resemble (though in simpler form) the more difficult Frobenius solutions to follow.

Example 29.2. Prove that the Cauchy-Euler equation has a regular singularity at $t = 0$.

Comparing with equation 29.1, we have that $a(t) = t^2$, $b(t) = \alpha t$, and $c(t) = \beta$. First, we observe that $a(0) = 0$, hence there is a singularity at $t = 0$. Then, applying the definition of a regular singularity (definition 29.1) with $t = 0$,

$$\lim_{t\to 0} \frac{(t)\cdot \alpha t}{t^2} = \alpha \tag{29.5}$$

$$\lim_{t\to 0} \frac{(t^2)\cdot \beta}{t^2} = \beta \tag{29.6}$$

Since both α and β are given real number, the limits exist and are finite. Hence by the definition of a regular singularity, there is a regular singularity at $t = 0$. $\square$

The following example demonstrates one of the problem that arises when we go blindly ahead and attempt to find a series solution to the Cauchy-Euler equation.

Example 29.3. Attempt to find a series solution about $t = 0$ to the Cauchy-Euler equation (29.4).

We begin by letting $y = \sum_{k=0}^{\infty} a_k t^k$ in (29.4), which gives

$$0 = t^2 \sum_{k=0}^{\infty} a_k k(k-1)t^{k-2} + \alpha t \sum_{k=0}^{\infty} k a_k t^{k-1} + \beta \sum_{k=0}^{\infty} a_k t^k \tag{29.7}$$

$$= \sum_{k=0}^{\infty} a_k k(k-1)t^k + \sum_{k=0}^{\infty} \alpha k a_k t^k + \sum_{k=0}^{\infty} \beta a_k t^k \tag{29.8}$$

$$= \sum_{k=0}^{\infty} a_k t^k [k(k-1) + \alpha k + \beta] \tag{29.9}$$

Since this must hold for all values of t, by linear independence we require that

$$a_k [k(k-1) + \alpha k + \beta] = 0 \tag{29.10}$$

for all $k = 0, 1, ...$

Hence either

$$a_k = 0 \tag{29.11}$$

or

$$k(k-1) + \alpha k + \beta = 0 \tag{29.12}$$

for all values of k. Since α and β are given real numbers, it is impossible for (29.12) to hold for all values of k because it is quadratic k. It will hold for at most two values of k, which are most likely not integers.

This leads us to the following conclusion: the only time when (29.4) has a series solution is when

$$r_{1,2} = \frac{1 - \alpha \pm \sqrt{(1-\alpha)^2 - 4\beta}}{2} \tag{29.13}$$

are both non-negative integers, say p and q, in which case the series solution is really just two terms

$$y = a_p t^p + a_q t^q \tag{29.14}$$

When there is not a non-negative integer solution to (29.13) then (29.4) does not have a seris solution. □

The result we found in example 29.3 does suggest to us one way to solve the Cauchy-Euler method. What if we relax the restriction on equation 29.10 that k be an integer. To see how this might come about, we consider a solution of the form

$$y = x^r \tag{29.15}$$

for some unknown number (integer, real, or possibly complex) r. To see what values of r might work, we substitute into the original ODE (29.4):

$$t^2r(r-1)t^{r-2} + \alpha trt^{r-1} + \beta t^r = 0 \tag{29.16}$$

$$\implies t^r[r(r-1) + \alpha r + \beta] = 0 \tag{29.17}$$

Since this must hold for all values of t, the second factor must be identically equal to zero. Thus we obtain

$$r(r-1) + \alpha r + \beta = 0 \tag{29.18}$$

without the restriction that $r \in \mathbb{Z}$. We call this the **indicial equation** (even though there are no indices) due to its relationship (and similarity) to the indicial equation that we will have to solve in Frobenius' method in chapter 30.

We will now consider each of the three possible cases for the roots

$$r_{1,2} = \frac{1-\alpha \pm \sqrt{(1-\alpha)^2 - 4\beta}}{2} \tag{29.19}$$

Case 1: Two real roots. If $r_1, r_2 \in \mathbb{R}$ and $r_1 \neq r_2$, then we have found two linearly independent solutions and the general solution is

$$y = C_1t^{r_1} + C_2t^{r_2} \tag{29.20}$$

Example 29.4. Find the general solution of

$$2t^2y'' + 3ty' - y = 0 \tag{29.21}$$

In standard from this becomes

$$t^2y'' + \frac{3}{2}ty' - \frac{1}{2}y = 0 \tag{29.22}$$

The indicial equation is

$$0 = r(r-1) - \frac{3}{2}r + \frac{1}{2} = \frac{2}{2}r^2 + \frac{1}{2}r + \frac{1}{2} \tag{29.23}$$

$$0 = 2r^2 + r + 1 = (2r-1)(r+1) \implies r = \frac{1}{2}, -1 \tag{29.24}$$

Hence

$$y = C_1\sqrt{t} + \frac{C_2}{t} \quad \square \tag{29.25}$$

Case 2: Two equal real roots. Suppose $r_1 = r_2 = r$. Then

$$r = \frac{1-\alpha}{2} \tag{29.26}$$

because the square root must be zero (ie., $(1-\alpha)^2 - 4\beta = 0$) to give a repeated root.

We have one solution is given by $y_1 = t^r = t^{(1-\alpha)/2}$. To get a second solution we can use reduction of order.

From Abel's formula the Wronskian is

$$W = C\exp\left(-\int \frac{\alpha t dt}{t^2}\right) = C\exp\left(-\alpha\int\frac{dt}{t}\right) = Ce^{-\alpha\ln t} = \frac{C}{t^\alpha} \tag{29.27}$$

By the definition of the Wronskian a second formula is given by

$$\begin{aligned} W &= y_1 y_2' - y_2 y_1' && (29.28)\\ &= \left(t^{(1-\alpha)/2}\right) y_2' - y_2'\left(\frac{1-\alpha}{2}\right)\left(t^{(-1-\alpha)/2}\right) && (29.29)\end{aligned}$$

Equating the two expressions for W,

$$\begin{aligned} \left(t^{(1-\alpha)/2}\right) y_2' - y_2'\left(\frac{1-\alpha}{2}\right)\left(t^{(-1-\alpha)/2}\right) &= \frac{C}{t^\alpha} && (29.30)\\ y_2' - \left(\frac{1-\alpha}{2}\right)\frac{t^{(-1-\alpha)/2}}{t^{(1-\alpha)/2}} y_2' &= \frac{C}{t^\alpha\left(t^{(1-\alpha)/2}\right)} && (29.31)\\ y_2' + \left(\frac{\alpha-1}{2t}\right) y_2' &= \frac{C}{t^{(1+\alpha)/2}} && (29.32)\end{aligned}$$

This is a first order linear equation; an integrating factor is

$$\mu(t) = \exp\left(\int\frac{\alpha-1}{2t}dt\right) = \exp\left(\frac{\alpha-1}{2}\ln t\right) = t^{(\alpha-1)/2} \tag{29.33}$$

If we denote $q(t) = Ct^{-(1+\alpha)/2}$ then the general solution of (29.32) is

$$\begin{aligned} y &= \frac{1}{\mu t}\left[\int \mu(t)q(t)dt + C_1\right] && (29.34)\\ &= t^{(1-\alpha)/2}\left[\int t^{(\alpha-1)/2} C t^{-(1+\alpha)/2} dt + C_1\right] && (29.35)\\ &= Cy_1\int\frac{dt}{t} + C_1 y_1 && (29.36)\\ &= Cy_1\ln|t| + C_1 y_1 && (29.37)\end{aligned}$$

because $y_1 = t^r = t^{(1-\alpha)/2}$.

Thus the second solution is $y_2 = y_1 \ln|t|$, and the the general solution to the Euler equation in case 2 becomes

$$y = C_1 t^r + C_2 t^r \ln|t| \quad \square \tag{29.38}$$

Example 29.5. Solve the Euler equation $t^2y'' + 5ty' + 4y = 0$.

The indicial equation is

$$0 = r(r-1) + 5r + 4 = r^2 + 4r + 4 = (r+2)^2 \implies r = -2 \tag{29.39}$$

This is case 2 with $r = -2$, so $y_1 = 1/t^2$ and $y_2 = (1/t^2)\ln|t|$, and the general solution is

$$y = \frac{C_1}{t^2} + \frac{C_2}{t^2}\ln|t| \quad \square \tag{29.40}$$

<u>Case 3: Complex Roots.</u> If the roots of $t^2y'' + \alpha ty' + \beta y = 0$ are a complex conjugate pair then we may denote them as

$$r = \lambda \pm i\mu \tag{29.41}$$

where λ and μ are real numbers. The two solutions are given by

$$\begin{aligned} t^r &= t^{\lambda \pm i\mu} = t^\lambda t^{\pm i\mu} && (29.42)\\ &= t^\lambda e^{\ln(t^{\pm i\mu})} && (29.43)\\ &= t^\lambda e^{\pm i\mu \ln t} && (29.44)\\ &= t^\lambda[\cos(\mu \ln t) \pm i \sin(\mu \ln t)] && (29.45) \end{aligned}$$

Hence the general solution is

$$\begin{aligned} y &= C_1 t^{r_1} + C_2 t^{r_2} && (29.46)\\ &= C_1 t^\lambda[\cos(\mu \ln t) + i\sin(\mu \ln t)] \\ &\qquad + C_2 t^\lambda[\cos(\mu \ln t) - i\sin(\mu \ln t)] && (29.47)\\ &= t^\lambda\left[(C_1 + C_2)\cos(\mu \ln t) + (C_1 - iC_2)\sin(\mu \ln t)\right] && (29.48) \end{aligned}$$

Thus for any C_1 and C_2 we can always find and A and B (and vice versa), where

$$\begin{aligned} A &= C_1 + C_2 && (29.49)\\ B &= C_1 - iC_2 && (29.50) \end{aligned}$$

so that

$$y = At^\lambda \cos(\mu \ln t) + Bt^\lambda \sin(\mu \ln t) \tag{29.51}$$

Example 29.6. Solve $t^2y'' + ty' + y = 0$. The indicial equation is

$$0 = r(r-1) + r + 1 = r^2 + 1 \implies r = \pm i \tag{29.52}$$

Thus the roots are a complex conjugate pair with real part $\lambda = 0$ and imaginary part $\mu = 1$. Hence the solution is

$$y = At^0 \cos(1 \cdot \ln t) + Bt^0 \sin(1 \cdot \ln t) = A \cos \ln t + B \sin \ln t \quad \square \tag{29.53}$$

Summary of Cauchy-Euler Equation

To solve $t^2y'' + \alpha ty' + \beta y = 0$ find the roots of

$$r(r-1) + \alpha r + \beta = 0 \tag{29.54}$$

There are three possible case.

1. If the roots are real and distinct, say $r_1 \neq r_2$, then the solution is

$$y = C_1 t^{r_1} + C_2 t^{r_2} \tag{29.55}$$

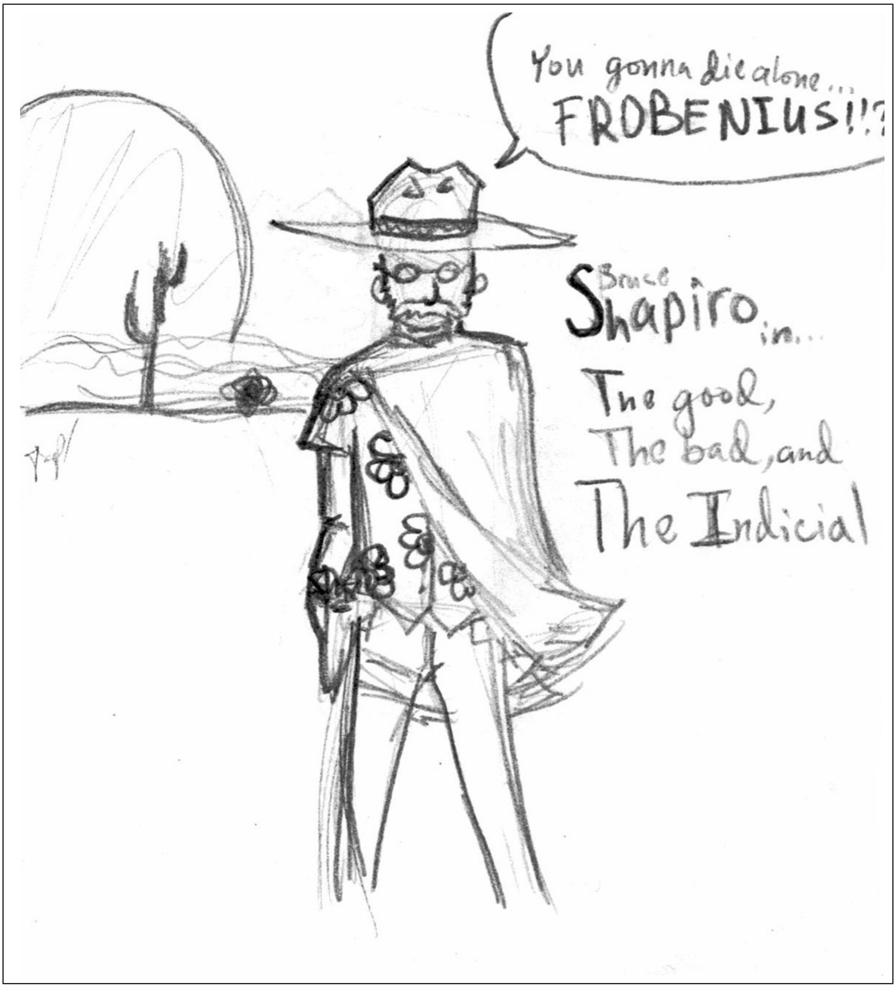
2. If the roots are real and equal, say $r = r_1 = r_2$, then the solution is

$$y = C_1 t^r + C_2 t^r \ln |t| \tag{29.56}$$

3. If the roots form a complex conjugate pair $r = \lambda \pm i\mu$, where $\lambda, \mu \in \mathbb{R}$, then the solution is

$$y = C_1 t^\lambda \cos(\mu \ln t) + C_2 t^\lambda \sin(\mu \ln t) \tag{29.57}$$

where, in each case, C_1 and C_2 are arbitrary constants (that may be determined by initial conditions).


You gonna die alone...
FROBENIUS!!?
Bruce
Shapiro in...
The good,
The bad, and
The Indicial

Lesson 30

The Method of Frobenius

If the equation

$$y'' + b(t)y' + c(t)y = 0 \tag{30.1}$$

has a regular singularity at the point $t = t_0$, then the functions

$$p(t) = (t - t_0)b(t) \tag{30.2}$$

$$q(t) = (t - t_0)^2 c(t) \tag{30.3}$$

are analytic at $t = t_0$ (see the discussion following (28.65)). Substituting (30.2) and (30.3) into (30.1) and then multiplying the result through by $(t - t_0)^2$ gives

$$(t - t_0)^2 y'' + (t - t_0)p(t)y' + q(t)y = 0 \tag{30.4}$$

Thus we are free to take equation (30.4) as the **canonical** form any second order differential equation with a regular singularity at $t = t_0$. By canonical form, we mean a standard form for describing any the properties of any second order differential equation with a regular singularity at $t = t_0$.

The simplest possible form that equation (30.4) can take occurs when $t_0 = 0$ and $p(t) = q(t) = 1$,

$$t^2 y'' + ty' + y = 0 \tag{30.5}$$

Equation (30.5) is a form of the Cauchy-Euler equation

$$t^2 y'' + aty' + by = 0, \tag{30.6}$$

where a and b are nonzero constants. The fact that theorem 28.1 fails to guarantee a series solution around $t = 0$ is illustrated by the following example.

Example 30.1. Attempt to find a series solution for equation (30.5).

Following our usual procedure we let $y = \sum_0^\infty c_k t^k$, differentiate twice, and substitute the results into the differential equation, leading to

$$\begin{aligned}
0 &= t^2 \sum_{k=0}^{\infty} c_k k(k-1)t^{k-2} + t\sum_{k=0}^{\infty} c_k k t^{k-1} + \sum_{k=0}^{\infty} c_k t^k && (30.7)\\
&= \sum_{k=0}^{\infty} c_k k(k-1)t^{k} + \sum_{k=0}^{\infty} c_k k t^{k} + \sum_{k=0}^{\infty} c_k t^k && (30.8)\\
&= \sum_{k=0}^{\infty} c_k (k(k-1)+k+1)t^{k} && (30.9)\\
&= \sum_{k=0}^{\infty} c_k (k^2+1)t^{k} && (30.10)
\end{aligned}$$

By linear independence,

$$c_k(k^2+1) = 0 \tag{30.11}$$

for all k. But since k^2+1 is a positive integer, this means $c_k = 0$ for all values of k. Hence the only series solution is the trivial one, $y = 0$. $\square$

Before one is tempted to conclude that there is no non-trivial solution to equation (30.5), consider the following example.

Example 30.2. Use the substitution $x = \ln t$ to find a solution to (30.5).

By the chain rule

$$y' = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{t}\frac{dy}{dx} \tag{30.12}$$

Differentiating a second time,

$$\begin{aligned}
y'' &= \frac{d}{dt}\left(\frac{1}{t}\frac{dy}{dx}\right) && (30.13)\\
&= -\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{d}{dt}\left(\frac{dy}{dx}\right) && (30.14)\\
&= -\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t}\frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{dt} && (30.15)\\
&= -\frac{1}{t^2}\frac{dy}{dx} + \frac{1}{t^2}\frac{d^2y}{dx^2} && (30.16)
\end{aligned}$$

Substituting (30.12) and (30.16) into (30.5) gives

$$\frac{d^2y}{dx^2} + y = 0 \tag{30.17}$$

This is a homogeneous linear equation with constant coefficients; the solution is

$$y = C_1 \cos x + C_2 \sin x \tag{30.18}$$
$$= C_1 \cos \ln t + C_2 \sin \ln t. \quad \square \tag{30.19}$$

The solution we found to (30.5) in example 30.2 is not even defined, much less analytic at $t = 0$. So there is no power series solution. However, it turns out that we can still make use of the result we found in example 30.1, which said that the power series solution only "exists" when $k^2 + 1 = 0$, if we drop the requirement that k be an integer, since then we would have $k = \pm i$, and our "series" solution would be

$$y = \sum_{k \in S} c_k t^k = c_{-i}t^{-i} + c_i t^i \tag{30.20}$$

where the sum is taken over the set $S = \{-i, i\}$.

Example 30.3. Show that the "series" solution given by (30.20) is equivalent to the solution we found in example 30.2.

Rewriting (30.20),

$$y = c_{-i}t^{-i} + c_i t^i \tag{30.21}$$
$$= c_{-i} \exp \ln t^{-i} + c_i \exp \ln t^i \tag{30.22}$$
$$= c_{-i} \exp(-i \ln t) + c_i \exp(i \ln t) \tag{30.23}$$
$$= c_{-1}[\cos(-i \ln t) + i \sin(-i \ln t)] + c_i[\cos(i \ln t) + i \sin(i \ln t)] \tag{30.24}$$
$$= c_{-1} \cos(i \ln t) - ic_{-1} \sin(i \ln t) + c_i \cos(i \ln t) + ic_i \sin(i \ln t) \tag{30.25}$$
$$= (c_i + c_{-1}) \cos(i \ln t) + i(c_i - c_{-1}) \sin(i \ln t) \tag{30.26}$$
$$= C_1 \cos(i \ln t) + C_2 \sin(i \ln t) \tag{30.27}$$

where

$$C_1 = c_{-i} + c_i \tag{30.28}$$
$$C_2 = i(c_i - c_{-i}) \tag{30.29}$$

Since this is a linear system and we can solve for c_i and c_{-i}, namely

$$c_i = \frac{1}{2}(C_1 - iC_2) \tag{30.30}$$
$$c_{-i} = \frac{1}{2}(C_1 + iC_2) \tag{30.31}$$

Given either solution, we can find constants that make it the same as the other solution, hence the two solutions are identical. $\square$

Since each term in (30.20) has the form t^α for some complex number α, this led Georg Ferdinand Frobenius to look for solutions of the form

$$y = (t - t_0)^\alpha S(t) \tag{30.32}$$

where $S(t)$ is analytic at t_0 and can be expanded in a power series,

$$S(t) = \sum_{k=0}^{\infty} c_k (t - t_0)^k \tag{30.33}$$

To determine the condition under which Frobenius' solution works, we differentiate (30.32) twice

$$y' = \alpha(t - t_0)^{\alpha-1} S + (t - t_0)^\alpha S' \tag{30.34}$$

$$y'' = \alpha(\alpha - 1)(t - t_0)^{\alpha-2} S + 2\alpha(t - t_0)^{\alpha-1} S' + (t - t_0)^\alpha S'' \tag{30.35}$$

and substitute equations (30.32), (30.34), and (30.35) into the differential equation (30.4):

$$\begin{aligned}
0 =& (t - t_0)^2 \left[\alpha(\alpha - 1)(t - t_0)^{\alpha-2} S + 2\alpha(t - t_0)^{\alpha-1} S' + (t - t_0)^\alpha S''\right] \\
&+ (t - t_0) \left[\alpha(t - t_0)^{\alpha-1} S + (t - t_0)^\alpha S'\right] p(t) + (t - t_0)^\alpha S q(t) \qquad (30.36) \\
=& (t - t_0)^{\alpha+2} S'' + [2\alpha + p(t)] (t - t_0)^{\alpha+1} S' \\
&+ [\alpha(\alpha - 1) + \alpha p(t) + q(t)] (t - t_0)^\alpha S \qquad (30.37)
\end{aligned}$$

For $t \neq t_0$, the common factor of $(t - t_0)^\alpha$ can be factored out to and the result becomes

$$0 = (t - t_0)^2 S'' + [2\alpha + p(t)] (t - t_0) S' + [\alpha(\alpha - 1) + \alpha p(t) + q(t)] S \tag{30.38}$$

Since $p(t)$, $q(t)$, and $S(t)$ are all analytic at t_0 the expression on the right-hand side of (30.38) is also analytic, and hence infinitely differentiable, at t_0. Thus it must be continuous at t_0 (differentiability implies continuity), and its limit as $t \to t_0$ must equal its value at $t = t_0$. Thus

$$[\alpha(\alpha - 1) + \alpha p(t_0) + q(t_0)] S(t_0) = 0 \tag{30.39}$$

So either $S(t_0) = 0$ (which means that $y(t_0) = 0$) or

$$\boxed{\alpha(\alpha - 1) + \alpha p(t_0) + q(t_0) = 0} \tag{30.40}$$

Equation (30.40) is called the **indicial equation** of the differential equation (30.4); if it is not satisfied, the Frobenius solution (30.32) will not work. The Indicial equation plays a role analogous to the characteristic equation but at regular singular points.

Before we prove that the indicial equation is also sufficient, we present the following example that demonstrates the **Method of Frobenius**.

Example 30.4. Find a Frobenius solution to **Bessel's equation** of order 1/2 near the origin,

$$t^2y'' + ty' + (t^2 - 1/4)y = 0 \tag{30.41}$$

By "near the origin" we mean "in a neighborhood that includes the point $t_0 = 0$.

The differential equation has the same form as $y'' + b(t)y' + c(t)y = 0$ with $b(t) = 1/t$ and $c(t) = (t^2 - 1/4)/t^2$, neither of which is analytic at $t = 0$. Thus the origin is not an ordinary point. However, since

$$p(t) = tb(t) = 1 \tag{30.42}$$

and

$$q(t) = t^2c(t) = t^2 - 1/4 \tag{30.43}$$

are both analytic at $t = 0$, we conclude that the singularity is regular. Letting $t_0 = 0$, we have $p(t_0) = p(0) = 1$ and $q(t_0) = -1/4$, so the indicial equation is

$$0 = \alpha(\alpha - 1) + \alpha - \frac{1}{4} = \alpha^2 - \frac{1}{4} \tag{30.44}$$

The roots are $\alpha = \pm 1/2$, so the two possible Frobenius solutions

$$y_1 = \sqrt{t}\sum_{k=0}^{\infty} a_k t^k = \sum_{k=0}^{\infty} a_k t^{k+1/2} \tag{30.45}$$

$$y_2 = \frac{1}{\sqrt{t}}\sum_{k=0}^{\infty} b_k t^k = \sum_{k=0}^{\infty} b_k t^{k-1/2} \tag{30.46}$$

Starting with the first solution,

$$y_1' = \sum_{k=0}^{\infty} a_k(k + 1/2)t^{k-1/2} \tag{30.47}$$

$$y_1'' = \sum_{k=0}^{\infty} a_k(k^2 - 1/4)t^{k-3/2} \tag{30.48}$$

Using (30.45) and (30.47) the differential equation gives

$$\begin{aligned} 0 = \sum_{k=0}^{\infty} a_k\left(k^2 - \frac{1}{4}\right)t^{k+1/2} + \sum_{k=0}^{\infty} a_k\left(k + \frac{1}{2}\right)t^{k+1/2} \\ + \left(t^2 - \frac{1}{4}\right)\sum_{k=0}^{\infty} a_k t^{k+1/2} \end{aligned} \tag{30.49}$$

Canceling the common factor of $\sqrt{t}$,

$$\begin{aligned}0 &= \sum_{k=0}^{\infty} a_k \left(k^2 - \frac{1}{4}\right) t^k + \sum_{k=0}^{\infty} a_k \left(k + \frac{1}{2}\right) t^k \\ &\qquad + \sum_{k=0}^{\infty} a_k t^{k+2} - \frac{1}{4} \sum_{k=0}^{\infty} a_k t^k && (30.50) \\ &= \sum_{k=0}^{\infty} a_k \left(k^2 - \frac{1}{4} + k + \frac{1}{2} - \frac{1}{4}\right) + \sum_{k=0}^{\infty} a_k t^{k+2} && (30.51) \\ &= \sum_{k=0}^{\infty} a_k (k^2 + k) + \sum_{k=0}^{\infty} a_k t^{k+2} && (30.52)\end{aligned}$$

Letting $j = k + 2$ in the second sum,

$$0 = \sum_{k=0}^{\infty} a_k (k^2 + k) t^k + \sum_{j=2}^{\infty} a_{j-2} t^j \tag{30.53}$$

Since the $k = 0$ term is zero and the $k = 1$ term is $2a_1 t$ in the first sum,

$$0 = 2a_1 t + \sum_{j=2}^{\infty} \left[a_j (j^2 + j) + a_{j-2}\right] t^j \tag{30.54}$$

By linear independence,

$$\begin{aligned}a_1 &= 0 && (30.55) \\ a_j &= \frac{-a_{j-2}}{j(j+1)},\ j \geq 2 && (30.56)\end{aligned}$$

Since $a_1 = 0$, all subsequent odd-numbered coefficients are zero (this follows from the second equation). Furthermore,

$$a_2 = \frac{-a_0}{3 \cdot 2}, a_4 = \frac{-a_2}{5 \cdot 4} = \frac{a_0}{5!}, a_6 = \frac{-a_4}{7 \cdot 6} = -\frac{a_0}{7!}, \cdots \tag{30.57}$$

Thus a Frobenius solution is

$$
\begin{aligned}
y_1 &= \sqrt{t}\sum_{k=0}^{\infty} a_k t^k && (30.58)\\
&= \sqrt{t}(a_0 + a_1 t + a_2 t^2 + \cdots) && (30.59)\\
&= a_0\sqrt{t}\left(1 - \frac{1}{3!}t^2 + \frac{1}{5!}t^4 - \frac{1}{7!}t^6 + \cdots\right) && (30.60)\\
&\frac{a_0}{\sqrt{t}}\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right) && (30.61)\\
&= \frac{a_0}{\sqrt{t}}\sin t && (30.62)
\end{aligned}
$$

Returning to the second solution

$$
y_2 = \frac{b_0}{\sqrt{t}} + b_1\sqrt{t} + b_2 t^{3/2} + b_3 t^{5/2} + \cdots \tag{30.63}
$$

This series cannot even converge at the origin unless $b_0 = 0$. This leads to

$$
\begin{aligned}
y_2 &= b_1\sqrt{t} + b_2 t^{3/2} + b_3 t^{5/2} + \cdots && (30.64)\\
&= \sqrt{t}(b_1 + b_2 t + b_3 t^2 + \cdots) && (30.65)
\end{aligned}
$$

which is the same as the first solution. So the Frobenius procedure, in this case, only gives us the one solution. □

Theorem 30.1. (Method of Frobenius) Let

$$
\begin{aligned}
p(t) &= \sum_{k=0}^{\infty} p_k (t - t_0)^k \text{ and} && (30.66)\\
q(t) &= \sum_{k=0}^{\infty} q_k (t - t_0)^k && (30.67)
\end{aligned}
$$

be analytic functions at $t = t_0$, with radii of convergence r, and let α_1, α_2 be the roots of the indicial equation

$$
\alpha(\alpha - 1) + \alpha p(t_0) + q(t_0) = 0. \tag{30.68}
$$

Then

1. If α_1, α_2 are both real and $\alpha = \max\{\alpha_1, \alpha_2\}$, there exists some set of constants $\{a_1, a_2, ...\}$ such that

$$
y = (t - t_0)^{\alpha}\sum_{k=0}^{\infty} a_k (t - t_0)^k \tag{30.69}
$$

is a solution of

$$(t-t_0)^2 y'' + (t-t_0)p(t)y' + q(t)y = 0 \tag{30.70}$$

2. If α_1, α_2 are both real and distinct, such that $\Delta = \alpha_1 - \alpha_2$ is not an integer, then (30.69) gives a second solution with $\alpha = \min\{\alpha_1, \alpha_2\}$and a (different) set of coefficients $\{a_1, a_2, ...\}$.

3. If α_1, α_2 are a complex conjugate pair, then there exists some sets of constants $\{a_1, a_2, ...\}$ and $\{b_1, b_2, ...\}$ such that

$$y_1 = (t-t_0)^{\alpha_1} \sum_{k=0}^{\infty} a_k (t-t_0)^k \tag{30.71}$$

$$y_2 = (t-t_0)^{\alpha_2} \sum_{k=0}^{\infty} b_k (t-t_0)^k \tag{30.72}$$

form a fundamental set of solutions to (30.70).

Example 30.5. Find the form of the Frobenius solutions for

$$2t^2 y'' + 7ty' + 3y = 0 \tag{30.73}$$

The differential equation can be put into the standard form

$$t^2 y'' + tp(t)y' + q(t)y = 0 \tag{30.74}$$

by setting $p(t) = 7/2$ and $q(t) = 3/2$. The indicial equation is

$$0 = \alpha^2 + (p_0 - 1)\alpha + q_0 \tag{30.75}$$

$$= \alpha^2 + (5/2)\alpha + 3/2 \tag{30.76}$$

$$= (\alpha + 3/2)(\alpha + 1) \tag{30.77}$$

Hence the roots are $\alpha_1 = -1$, $\alpha_2 = -3/2$. Since $\Delta = \alpha_1 - \alpha_2 = 1/2 \notin \mathbb{Z}$, each root leads to a solution:

$$y_1 = \frac{1}{t} \sum_{k=0}^{\infty} a_k t^k = \frac{a_0}{t} + a_1 + a_2 t + a_3 t^2 + \cdots \tag{30.78}$$

$$y_2 = t^{-3/2} \sum_{k=0}^{\infty} b_k t^k = \frac{b_0}{t^{3/2}} + \frac{b_1}{t^{1/2}} + b_2 t^{1/2} + b_3 t^{3/2} + \cdots \quad \square \tag{30.79}$$

Example 30.6. Find the form of the Frobenius solutions to

$$t^2y'' - ty + 2y = 0 \tag{30.80}$$

This equation can be written in the form $t^2y'' + tp(t)y' + q(t)y = 0$ where $p(t) = -1$ and $q(t) = 2$. Hence the indicial equation is

$$0 = \alpha^2 + (p_0 - 1)\alpha + q_0 = \alpha^2 - 2\alpha + 2 \tag{30.81}$$

The roots of the indicial equation are $\alpha = 1 \pm i$, a complex conjugate pair. Since $\Delta\alpha = (1+i) - (1-i) = 2i \notin \mathbb{Z}$, each root gives a Frobenius solution:

$$\begin{aligned} y_1 &= t^{1+i}\sum_{k=0}^{\infty} a_k t^k && (30.82)\\ &= (\cos\ln t + i\sin\ln t)\left(a_0t + a_1t^2 + a_2t^3 + \cdots\right) && (30.83)\\ y_2 &= t^{1-i}\sum_{k=0}^{\infty} b_k t^k && (30.84)\\ &= (\cos\ln t - i\sin\ln t)\left(b_0t + b_1t^2 + b_2t^3 + \cdots\right). \quad \square && (30.85) \end{aligned}$$

Example 30.7. Find the form of the Frobenius solution for the Bessel equation of order -3,

$$t^2y'' + ty' + (t^2 + 9) = 0 \tag{30.86}$$

This has $p(t) = 1$ and $q(t) = t^2 + 9$, so that $p_0 = 1$ and $q_0 = 9$. The indicial equation is

$$\alpha^2 + 9 = 0 \tag{30.87}$$

The roots are the complex conjugate pair $\alpha_1 = 3i, \quad \alpha_2 = -3i$. For a complex conjugate pair, each root gives a Frobenius solution, and hence we have two solutions

$$y_1 = t^{3i}\sum_{k=0}^{\infty} a_k t^k = [\cos(3\ln t) + i\sin(3\ln t)]\sum_{k=0}^{\infty} a_k t^k \tag{30.88}$$

$$y_2 = t^{-3i}\sum_{k=0}^{\infty} b_k t^k = [\cos(3\ln t) - i\sin(3\ln t)]\sum_{k=0}^{\infty} b_k t^k \tag{30.89}$$

where we have used the identity

$$t^{ix} = e^{ix\ln t} = \cos(x\ln t) + i\sin(x\ln t) \tag{30.90}$$

in the second form of each expression. $\square$

Proof. (of theorem 30.1). Suppose that

$$y = (t - t_0)^\alpha S(x) \tag{30.91}$$

is a solution of (30.70), where

$$S = \sum_{n=0}^{\infty} c_n (t - t_0)^n. \tag{30.92}$$

We will derive formulas for the c_k.

For $n = 1$, we start by differentiating equation (30.38)

$$\begin{aligned} 0 =& 2(t - t_0)S'' + (t - t_0)^2 S''' \\ &+ [p(t) + 2\alpha]\, S' + (t - t_0)p'(t)S' + (t - t_0)\,[p(t) + 2\alpha]\, S''' \\ &+ [q'(t) + \alpha p'(t)]\, S + [q(t) + \alpha(\alpha - 1) + \alpha p(t)]\, S' \end{aligned} \tag{30.93}$$

Everything on the right hand side of this equation is analytic, and hence continuous. Taking the limit as $t \to t_0$, we have,

$$\begin{aligned} 0 =& [p(t_0) + 2\alpha]\, S'(t_0) \\ &+ [q'(t_0) + \alpha p'(t_0)]\, S + [q(t_0) + \alpha(\alpha - 1) + \alpha p(t_0)]\, S'(t_0) \end{aligned} \tag{30.94}$$

By Taylor's Theorem $c_n = S^{(n)}(t_0)/n!$ so that $c_0 = S(t_0)$ and $c_1 = S'(t_0)$; similarly, $p(t_0) = p_0$, $p'(t_0) = p_1$, $q(t_0) = q_0$, and $q'(t_0) = q_1$ in (30.66), and therefore

$$0 = (p_0 + 2\alpha)c_1 + [q_1 + \alpha p_1]c_0 + [q_0 + \alpha(\alpha - 1) + \alpha p_0]c_1 \tag{30.95}$$

The third term is zero (this follows from (30.68)) and hence

$$c_1 = -\frac{q_1 + \alpha p_1}{p_0 + 2\alpha} c_0. \tag{30.96}$$

For $n > 1$, we will need to differentiate equation (30.38) n times, to obtain a recursion relationship between the remaining coefficients, starting with

$$\begin{aligned} 0 = D^{(n)}\left[(t - t_0)^2 S''\right] &+ D^{(n)}\left\{[2\alpha + p(t)]\,(t - t_0)S'\right\} \\ &+ D^{(n)}\left\{[\alpha(\alpha - 1) + \alpha p(t) + q(t)]\, S\right\} \end{aligned} \tag{30.97}$$

We then apply the identity

$$D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} (D^k u)(D^{n-k} v) \tag{30.98}$$

term by term to (30.97). Starting with the first term,

$$D^n\left[(t-t_0)^2S''\right] = \sum_{j=0}^{n}\binom{n}{j}\left[D^j(t-t_0)^2\right]\left[D^{n-j}S''\right] \tag{30.99}$$

$$= \binom{n}{0}(t-t_0)^2D^nS'' + 2\binom{n}{1}(t-t_0)D^{n-1}S'' + 2\binom{n}{2}D^{n-2}S'' \tag{30.100}$$

Even for $n>2$ there are only 3 terms because $D^n(t-t_0)^2=0$ for $n\geq 3$. Hence

$$D^n\left[(t-t_0)^2S''\right] = (t-t_0)^2S^{(n+2)} + 2(n-1)(t-t_0)S^{(n+1)} + n(n-1)S^{(n)} \tag{30.101}$$

At $t=t_0$,

$$D^n\left[(t-t_0)^2S''\right]\Big|_{t=t_0} = n(n-1)S^{(n)}(t_0) = n(n-1)c_n n! \tag{30.102}$$

Similarly, the second term in (30.97)

$$\begin{aligned} &D^n\left\{(t-t_0)[2\alpha+p(t)]S'(t)\right\} \\ &\quad= \sum_{k=0}^{n}\binom{n}{k}D^k(t-t_0)D^{n-k}\left\{S'(t)[2\alpha+p(t)]\right\} && (30.103)\\ &\quad= (t-t_0)D^n\left\{S'(t)[2\alpha+p(t)]\right\} + nD^{n-1}\left\{S'(t)[2\alpha+p(t)]\right\} && (30.104)\end{aligned}$$

At $t=t_0$,

$$\begin{aligned} &D^n\left\{(t-t_0)[2\alpha_- + p(t)]S'(t)\right\}(t_0) \\ &\quad= nD^{n-1}\left\{S'(t)[2\alpha+p(t)]\right\}(t_0) && (30.105)\\ &\quad= n\sum_{k=0}^{n-1}\binom{n-1}{k}\left\{[2\alpha+p(t)]^{(k)}(t_0)\right\}S^{(n-k)}(t_0) && (30.106)\\ &\quad= n[2\alpha+p(t_0)]S^{(n)}(t_0) \\ &\qquad + n\sum_{k=1}^{n-1}\binom{n-1}{k}\left\{[2\alpha+p(t)]^{(k)}(t_0)\right\}S^{(n-k)}(t_0) && (30.107)\\ &\quad= n[2\alpha+p(t_0)]S^{(n)}(t_0) + n\sum_{k=1}^{n-1}\binom{n-1}{k}p^{(k)}(t_0)S^{(n-k)}(t_0) && (30.108)\end{aligned}$$

By Taylor's theorem $p^{(k)}(t_0) = k!p_k$ and $S^{(n-k)}(t_0) = (n-k)!c_{n-k}$, where p_k and c_k are the Taylor coefficients of $p(t)$ and $S(t)$, respectively,

Similarly, the third term in (30.97) is

$$\begin{aligned} &D^{(n)}\left\{[\alpha(\alpha-1)+\alpha p(t)+q(t)]\,S\right\}(t_0) \\ &\quad = \sum_{k=0}^{n}\binom{n}{k}\left\{[\alpha(\alpha-1)+\alpha p(t)+q(t)]^{(k)}(t_0)\right\}S^{(n-k)}(t_0) \end{aligned} \tag{30.109}$$

Extracting the first $(k=0)$ term,

$$\begin{aligned} &D^{(n)}\left\{[\alpha(\alpha-1)+\alpha p(t)+q(t)]\,S\right\}(t_0) \\ &\quad = [\alpha(\alpha-1)+\alpha p_0+q_0]\,S^{(n)}(t_0) \\ &\qquad + \sum_{k=1}^{n}\binom{n}{k}\left[\alpha p^{(k)}(t_0)+q^{(k)}(t_0)\right]S^{(n-k)}(t_0) \end{aligned} \tag{30.110}$$

By the indicial equation, the first term is zero. Substituting the formulas for the Taylor coefficients and simplifying,

$$D^{(n)}\left\{[\alpha(\alpha-1)+\alpha p(t)+q(t)]\,S\right\}(t_0) = \sum_{k=1}^{n}\binom{n}{k}\left[\alpha k!p_k+k!q_k\right](n-k)!c_{n-k} \tag{30.111}$$

Substituting (30.111) and (30.102) into (30.97),

$$\begin{aligned} 0 =& n(n-1)c_n n! + nn!(2\alpha+p_0)c_n \\ &+ \sum_{k=1}^{n-1} n!(n-k)p_k c_{n-k} + \sum_{k=1}^{n} n!\left[\alpha p_k+q_k\right]c_{n-k} \end{aligned} \tag{30.112}$$

Rearranging and simplifying,

$$n(n-1+2\alpha+p_0)c_n + \sum_{k=1}^{n-1}[(n-k+\alpha)p_k+q_k]c_{n-k} = 0 \tag{30.113}$$

Letting $r>0$ be the radius of convergence of the Taylor series for p and q, then there is some number M such that $|p_k|\,r^k \le M$ and $|q_k|\,r^k \le M$. Then

$$|n(n-1+2\alpha+p_0)c_n| \le M\sum_{k=1}^{n-1}\frac{1}{r^k}\left[|n-k+\alpha|+1\right]|c_{n-k}| \tag{30.114}$$

Define the numbers $C_0 = |c_0|$, $C_1 = |c_1|$, and, for $n \ge 2$,

$$|n-1+2\alpha+p_0|\,C_n = \frac{M}{n}\sum_{k=1}^{n-1}\frac{1}{r^k}\left[|n-k+\alpha|+1\right]C_{n-k} \tag{30.115}$$

Let $j = n - k$. Then

$$|n-1+2\alpha+p_0|\,C_n = \frac{M}{nr^n}\sum_{j=1}^{n-1} r^j\left[|j+\alpha|+1\right]C_j \tag{30.116}$$

Evaluating (30.116) for $n+1$,

$$|n+2\alpha+p_0|\,C_{n+1} = \tfrac{M}{(n+1)r^{n+1}}\textstyle\sum_{j=1}^{n} r^j\left[|j+\alpha|+1\right]C_j \tag{30.117}$$

Therefore

$$|n+2\alpha+p_0|\,C_{n+1}r(n+1) = \{n\,|n-1+2\alpha+p_0| + M\left[|n+\alpha|+1\right]\}\,C_n \tag{30.118}$$

so that

$$\begin{aligned}\frac{C_{n+1}(t-t_0)}{C_n} &= \frac{n\,|n-1+2\alpha+p_0| + M\left[|n+\alpha|+1\right]}{|n+2\alpha+p_0|\,(n+1)}\,\frac{(t-t_0)}{r} && (30.119)\\ &= \frac{\frac{n|n-1+2\alpha+p_0|}{n^2} + M\frac{|n+\alpha|+1}{n^2}}{\frac{|n+2\alpha+p_0|(n+1)}{n^2}}\,\frac{(t-t_0)}{r} && (30.120)\end{aligned}$$

Hence

$$\lim_{n\to\infty}\left|\frac{C_{n+1}(t-t_0)}{C_n}\right| = \left|\frac{t-t_0}{r}\right| < 1 \tag{30.121}$$

Therefore by the ratio test $\sum_{k=0}^{\infty} C_k(t-t_0)^k$ converges; by the comparison test, the sum $\sum_{n=0}^{\infty} c_n(t-t_0)^n$ also converges. Equations (30.96) and (30.113) give formulas for the c_n

$$c_1 = -\frac{q_1+\alpha p_1}{p_0+2\alpha}c_0 \tag{30.122}$$

and

$$c_n = \frac{-1}{n(n-1+2\alpha+p_0)}\sum_{k=1}^{n-1}\left[(n-k+\alpha)p_k+q_k\right]c_{n-k} \tag{30.123}$$

so that

$$y = (t-t_0)^\alpha\sum_{n=0}^{\infty} c_n(t-t_0)^n \tag{30.124}$$

is a solution of (30.70) so long as

$$n-1+2\alpha+p_0 \neq 0. \tag{30.125}$$

By the indicial equation, $\alpha^2 + \alpha(p_0 - 1) + q_0 = 0$, so that

$$\alpha = \frac{1}{2}\left[1 - p_0 \pm \sqrt{(1-p_0)^2 - 4q_0}\right] \tag{30.126}$$

Designate the two roots by α_+ and α_- and define

$$\Delta = \alpha_+ - \alpha_- = \sqrt{(1-p_0)^2 - 4q_0} \tag{30.127}$$

If they are real, then the larger root satisfies $2\alpha_+ \geq 1 - p_0$, so that

$$n - 1 + 2\alpha_+ + p_0 \geq n - 1 + 1 - p_0 + p_0 = n > 0 \tag{30.128}$$

Therefore (30.124) gives a solution for the larger of the two roots. The other root gives

$$n - 1 + 2\alpha_- + p_0 \quad = \quad n - 1 + 1 - p_0 - \sqrt{(p_0-1)^2 - 4q_0} + p_0 \tag{30.129}$$

This is never zero unless the two roots differ by an integer. Thus the smaller root also gives a solution, so long as the roots are different and do not differ by an integer.

If the roots are complex, then

$$\alpha = \frac{1}{2}\left[1 - p_0 \pm i\Delta\right] \tag{30.130}$$

where $\Delta \neq 0$ and therefore

$$n - 1 + 2\alpha + p_0 = n - 1 + 1 - p_0 \pm i\Delta + p_0 = n \pm i\Delta \tag{30.131}$$

which can never be zero. Hence both complex roots lead to solutions. $\square$

When the difference between the two roots of the indicial equation is an integer, a second solution can be found by reduction of order.

Theorem 30.2. (Second Frobenius solution.) Suppose that $p(t)$ and $q(t)$ are both analytic with some radius of convergence r, and that $\alpha_1 \geq \alpha_2$ are two (possibly distinct) real roots of the indicial equation

$$\alpha^2 + (p_0 - 1)\alpha + q_0 = 0 \tag{30.132}$$

and let $\Delta = \alpha_1 - \alpha_2$. Then for some set of constants $\{c_0, c_1, ...\}$

$$y_1(t) = (t - t_0)^{\alpha_1} \sum_{k=0}^{\infty} c_k (t - t_0)^k \tag{30.133}$$

is a solution of

$$(t-t_0)^2y'' + (t-t_0)p(t)y' + q(t)y = 0 \tag{30.134}$$

with radius of convergence r (as was shown in theorem 30.1), and a second linearly independent solution, also with radius of convergence r, is given by one of the following three cases.

1. If $\alpha_1 = \alpha_2 = \alpha$, then

$$y_2 = ay_1(t)\ln|t-t_0| + (t-t_0)^\alpha \sum_{k=0}^{\infty} a_k(t-t_0)^k \tag{30.135}$$

2. If $\Delta \in \mathbb{Z}$, then

$$y_2 = ay_1(t)\ln|t-t_0| + (t-t_0)^{\alpha_2} \sum_{k=0}^{\infty} a_k(t-t_0)^k \tag{30.136}$$

3. If $\Delta \notin \mathbb{Z}$, then

$$y_2 = (t-t_0)^{\alpha_2} \sum_{k=0}^{\infty} a_k(t-t_0)^k \tag{30.137}$$

for some set of constants $\{a, a_0, a_1, ...\}$.

Proof. We only give the proof for $t_0 = 0$; otherwise, make the change of variables to $x = t - t_0$ and the proof is identical. Then the differential equation becomes

$$t^2y'' + tp(t)y' + q(t)y = 0 \tag{30.138}$$

and the first Frobenius solution (30.133) is

$$u(t) = t^\alpha \sum_{k=0}^{\infty} c_k t^k \tag{30.139}$$

where α is the larger of the two roots of the indicial equation. By Abel's formula the Wronskian of (30.138)

$$W(t) = \exp\left\{-\int \frac{p(t)}{t}dt\right\} = \exp\left\{-\sum_{k=0}^{\infty} p_k \int t^{k-1}dt\right\} \tag{30.140}$$

where $\{p_0, p_1, ...\}$ are the Taylor coefficients of $p(t)$. Integrating term by term

$$W(t) = \exp\left\{-p_0\ln|t| - \sum_{k=1}^{\infty}\frac{p_k}{k}t^k\right\} = |t|^{-p_0}\exp\left\{-\sum_{k=1}^{\infty}\frac{p_k}{k}t^k\right\} \tag{30.141}$$

Since

$$e^{-u} = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{u^k}{k!} \tag{30.142}$$

we can write

$$\exp\left\{ -\sum_{k=1}^{\infty} \frac{p_k}{k} t^k \right\} = 1 + \sum_{k=1}^{\infty} a_k t^k \tag{30.143}$$

for some sequence of numbers $a_1, a_2, \ldots$. We do not actually need to know these numbers, only that they exist. Hence

$$W(t) = |t|^{-p_0} \left\{ 1 + \sum_{k=1}^{\infty} a_k t^k \right\} \tag{30.144}$$

By the method of reduction of order, a second solution is given by

$$v(t) = u(t) \int \frac{W(t)}{u^2(t)} dt = u(t) \int \frac{|t|^{-p_0}}{u^2(t)} \left\{ 1 + \sum_{k=1}^{\infty} a_k t^k \right\} dt \tag{30.145}$$

From equation (30.139), since u is analytic, so is $1/u$, except possibly at its zeroes, so that $1/u$ can also be expanded in a Taylor series. Thus

$$\frac{1}{u^2(t)} = t^{-2\alpha} \left\{ \sum_{k=0}^{\infty} c_k t^k \right\}^{-2} = t^{-2\alpha} c_0^{-2} \left\{ 1 + \sum_{k=1}^{\infty} b_k t^k \right\} \tag{30.146}$$

for some sequence $b_1, b_2, \ldots$ Letting $K = 1/c_0^2$,

$$v(t) = K u(t) \int |t|^{-p_0 - 2\alpha} \left\{ 1 + \sum_{k=1}^{\infty} b_k t^k \right\} \left\{ 1 + \sum_{k=1}^{\infty} a_k t^k \right\} dt \tag{30.147}$$

for some sequence $d_1, d_2, \ldots$

By the quadratic formula

$$\alpha_1 = \frac{1}{2} \left(1 - p_0 + \sqrt{(1 - p_0)^2 - 4 q_0} \right) = \frac{1}{2} (1 - p_0 + \Delta) \tag{30.148}$$

$$\alpha_2 = \frac{1}{2} \left(1 - p_0 - \sqrt{(1 - p_0)^2 - 4 q_0} \right) = \frac{1}{2} (1 - p_0 - \Delta) \tag{30.149}$$

and therefore

$$2\alpha + p_0 = 1 + \Delta \tag{30.150}$$

since we have chosen $\alpha = \max(\alpha_1, \alpha_2) = \alpha_1$. Therefore

$$v(t) = K u(t) \int |t|^{-(1+\Delta)} \left\{ 1 + \sum_{k=1}^{\infty} d_k t^k \right\} dt \tag{30.151}$$

Evaluation of the integral depends on the value of Δ. If $\Delta = 0$ the first term is logarithmic; if $\Delta \in \mathbb{Z}$ then the $k = \Delta$ term in the sum is logarithmic; and if $\Delta \notin \mathbb{Z}$, there are no logarithmic terms. We assume in the following that $t > 0$, so that we can set $|t| = t$; the $t < 0$ case is left as an exercise.

Case 1. $\Delta = 0$. In this case equation (30.151) becomes

$$v(t) = Ku(t)\left\{\int \frac{1}{t}dt + \sum_{k=1}^{\infty} d_k \int t^{k-1}dt\right\} \tag{30.152}$$

$$= Ku(t)\left\{\ln|t| + \sum_{k=1}^{\infty} d_k \frac{t^k}{k}\right\} \tag{30.153}$$

Substitution of equation (30.139) gives

$$v(t) = Kt^{\alpha}\sum_{k=0}^{\infty} c_k t^k \left[\ln|t| + \sum_{j=1}^{\infty} d_j \frac{t^j}{j}\right] \tag{30.154}$$

$$= Kt^{\alpha}\ln|t|\sum_{k=0}^{\infty} c_k t^k + Kt^{\alpha}\sum_{k=0}^{\infty}\sum_{j=1}^{\infty} c_k d_j \frac{t^{j+k}}{j} \tag{30.155}$$

Since the product of two differentiable functions is differentiable, then the product of two analytic functions is analytic, hence the product of two power series is also a power series. The last term above is the product of two power series, which we can re-write as a single power series with coefficients e_j as follows,

$$v(t) = Kt^{\alpha}\ln|t|\sum_{k=0}^{\infty} c_k t^k + Kt^{\alpha}\sum_{k=0}^{\infty} e_k t^k \tag{30.156}$$

$$= K\ln|t|\,u(t) + t^{\alpha}\sum_{k=0}^{\infty} e_k t^k \tag{30.157}$$

for some sequence of numbers $\{e_0, e_1, ...\}$. This proves equation 30.135.

Case 2. $\Delta =$ a positive integer. In this case we rewrite (30.151) as

$$v(t) = Ku(t)\sum_{k=0}^{\infty} d_k \int t^{k-(1+\Delta)}dt \tag{30.158}$$

Integrating term by term,

$$v(t) = Ku(t)\left[d_\Delta \int t^{-1}dt + \sum_{k=0,k\neq\Delta}^{\infty} d_k \int t^{k-(1+\Delta)}dt\right] \tag{30.159}$$

$$= Ku(t)\left[d_\Delta \ln|t| + \sum_{k=0,k\neq\Delta}^{\infty} \frac{d_k}{k-\Delta}t^{k-\Delta}\right] \tag{30.160}$$

Substituting equation (30.139) in the second term,

$$v(t) = au(t)\ln|t| + Kt^\alpha \sum_{k=0}^{\infty} c_k t^k \sum_{k=0,k\neq\Delta}^{\infty} \frac{d_k}{k-\Delta}t^{k-\Delta} \tag{30.161}$$

where a is a constant, and we have factored out the common $t^{-\Delta}$ in the second line.

Since the product of two power series is a power series, then there exists a sequence of numbers $\{a_0, a_1, ...\}$ such that

$$v(t) = au(t)\ln|t| + t^{\alpha_2}\sum_{k=0}^{\infty} a_k t^k \tag{30.162}$$

where we have used the fact that $\alpha_2 = \alpha - \Delta$. This proves (30.136).

Case 3. $\Delta \neq 0$ and Δ is not an integer. Integrating (30.151) term by term,

$$v(t) = Ku(t)\left[-\frac{t^{-\Delta}}{\Delta} + \sum_{k=1}^{\infty} \frac{d_k}{k-\Delta}t^{k-\Delta}\right] \tag{30.163}$$

$$= u(t)t^{-\Delta}\sum_{k=0}^{\infty} f_k t^k \tag{30.164}$$

where $f_0 = -K/\Delta$ and $f_k = Kd_k/(k-\Delta)$, for $k = 1, 2, ...$ Substitution for $u(t)$ gives

$$v(t) = t^{\alpha-\Delta}\sum_{k=0}^{\infty} c_k t^k \sum_{j=0}^{\infty} f_j t^j \tag{30.165}$$

Since $\alpha_2 = \alpha - \Delta$, and since the product of two power series is a power series, there exists some sequence of constants $\{a_0, a_1, ...\}$ such that

$$v(t) = t^{\alpha_2}\sum_{k=0}^{\infty} a_k t^k \tag{30.166}$$

which proves (30.137). $\square$

Example 30.8. In example 30.4 we found that one solution of Bessel's equation of order 1/2, given by

$$t^2y'' + ty' + (t^2 - 1/4)y = 0 \tag{30.167}$$

near the origin is

$$y_1 = \frac{\sin t}{\sqrt{t}} \tag{30.168}$$

Find a second solution using the method of Frobenius.

In example 30.4 we found that the the roots of the indicial equation are

$$\alpha_1 = 1/2 \text{ and } \alpha_2 = -1/2 \tag{30.169}$$

Since the difference between the two roots is

$$\Delta = \alpha_1 - \alpha_2 = 1 \tag{30.170}$$

We find the result from theorem 30.2, case 2, which gives a second solution

$$y_2(t) = ay_1(t)\ln|t| + t^{\alpha_2}\sum_{k=0}^{\infty} a_k t^k \tag{30.171}$$

The numbers a and $a_0, a_1, ...$ are found by substituting (30.171) into the original differential equation and using linear independence. In fact, since we have a neat, closed form for the first solution that is not a power series, it is easier to find the second solution directly by reduction of order.

In standard form the differential equation can be rewritten as

$$y'' + \frac{1}{t}y' + \frac{t^2 - 1/4}{t^2}y = 0 \tag{30.172}$$

which has the form of $y'' + py' + q = 0$ with $p(t) = 1/y$. By Abel's formula, one expression for the Wronskian is

$$W = C\exp\int \frac{-1}{t}dt = \frac{C}{t} \tag{30.173}$$

According to the reduction of order formula, a second solution is given by

$$y_2 = y_1(t)\int \frac{W(t)}{y_1(t)^2}dt = \frac{\sin t}{\sqrt{t}}\int \frac{1}{t}\cdot\left(\frac{\sqrt{t}}{\sin t}\right)^2 dt = \frac{\sin t}{\sqrt{t}}\int \csc^2 t \tag{30.174}$$

$$= -\frac{\sin t}{\sqrt{t}}\cot t = -\frac{\cos t}{\sqrt{t}} \qquad \square \tag{30.175}$$

Example 30.9. Find the form of the Frobenius solutions to Bessel's equation of order 3,

$$t^2y'' + ty + (t^2 - 9)y = 0 \tag{30.176}$$

Equation (30.176) has the form $t^2y'' + tp(t)y' + q(t)y = 0$, where $p(t) = 1$ and $q(t) = t^2 - 9$ are both analytic functions at $t = 0$. Hence $p_0 = 1$, $q_0 = -9$, and the indicial equation $\alpha^2 + (p_0 - 1)\alpha + q_0 = 0$ is

$$\alpha^2 - 9 = 0 \tag{30.177}$$

Therefore $\alpha_1 = 3$ and $\alpha_2 = -3$. The first Frobenius solution is

$$y_1 = t^3 \sum_{k=0}^{\infty} a_k t^k = a_0 t^3 + a_1 t^4 + a_2 t^5 + \cdots \tag{30.178}$$

Since $\Delta = \alpha_1 - \alpha_2 = 6 \in \mathbb{Z}$, the second solution is

$$y_2 = a\,(\ln t)\, t^3 \sum_{k=0}^{\infty} a_k t^k + t^{-3} \sum_{k=0}^{\infty} b_k t^k \tag{30.179}$$

The coefficients a, a_k and b_k are found by substituting the expressions (30.178) and (30.179) into the differential equation. $\square$

Lesson 31

Linear Systems

The general linear system of order n can be written as

$$\left.\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n} + f_1(t) \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn} + f_2(t) \end{aligned}\right\} \tag{31.1}$$

where the a_{ij} are either constants (for systems with constant coefficients) or depend only on t and the functions $f_i(t)$ are either all zero (for homogeneous systems) or depend at most on t.

We typically write this a matrix equation

$$\begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix} \tag{31.2}$$

We will write this as the matrix system

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{f} \tag{31.3}$$

where it is convenient to think of $\mathbf{y}, \mathbf{f}(t) : \mathbb{R} \to \mathbb{R}^n$ as vector-valued functions.

In analogy to the scalar case, we call a set of solutions to the vector equation $\{\mathbf{y}_1, ..., \mathbf{y}_n\}$ to the homogeneous equation

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \tag{31.4}$$

a **fundamental set of solutions** if every solution to (31.4) can be written in the form

$$\mathbf{y} = C_1\mathbf{y}_1(t) + \cdots + C_n\mathbf{y}_n(t) \tag{31.5}$$

for some set of constants $C_1, \dots, C_n$. We define the **fundamental matrix** as

$$\mathbf{W} = \begin{pmatrix} \mathbf{y}_1 & \cdots & \mathbf{y}_n \end{pmatrix} \tag{31.6}$$

and the **Wronskian** as its determinant,

$$W(t) = \det \mathbf{W} \tag{31.7}$$

The columns of the fundamental matrix contain the vector-valued solutions, not a solution and its derivatives, as they did for the scalar equation.

However, this should not be surprising, because if we convert an nth order equation into a system by making the change of variables $u_1 = y$, $u_2 = y'$, $u_3 = y''$, ..., $u_n = y^{(n-1)}$, the two representations will be identical.

Homogeneous Systems

Theorem 31.1. The fundamental matrix of the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ satisfies the differential equation.

Proof. Let $\mathbf{y}_1$, ..., $\mathbf{y}_n$ be a fundamental set of solutions. Then each solution satisfies $\mathbf{y}_i' = \mathbf{A}\mathbf{y}_i$. But by definition of the fundamental matrix,

$$\begin{aligned} \mathbf{W}' &= \begin{pmatrix} \mathbf{y}_1' & \cdots & \mathbf{y}_n' \end{pmatrix} && (31.8)\\ &= \begin{pmatrix} \mathbf{A}\mathbf{y}_1 & \cdots & \mathbf{A}]\mathbf{y}_n \end{pmatrix} && (31.9)\\ &= \mathbf{A}\begin{pmatrix} \mathbf{y}_1 & \cdots & \mathbf{y}_n \end{pmatrix} && (31.10)\\ &= \mathbf{A}\mathbf{W} && (31.11) \end{aligned}$$

Thus $\mathbf{W}$ satisfies the same differential equation. $\square$

For $n = 2$, we can write equation the homogeneous system as

$$\left.\begin{aligned} u' &= au + bv \\ v' &= cu + dv \end{aligned}\right\} \tag{31.12}$$

where $u = y_1$, $v = y_2$, and a, b, c, and d are all real constants.

the characteristic equation of the matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{31.13}$$

is

$$\lambda^2 - T\lambda + \Delta = 0 \tag{31.14}$$

where

$$\left.\begin{aligned} T &= a + d = \mathrm{trace}(\mathbf{A}) \\ \Delta &= ad - bc = \det(\mathbf{A}) \end{aligned}\right\} \tag{31.15}$$

The roots λ_1 and λ_2 of (31.14) are the eigenvalues of $\mathbf{A}$; we will make use of this fact shortly.

By rearranging the system (31.12) it is possible to separate the two first order equations into equivalent second order equations with the variables separated.

<u>If both $b = 0$ and $c = 0$</u>, we have two completely independent first order equations,

$$u' = au, \quad v' = dv \tag{31.16}$$

If $b \neq 0$, we can solve the first of equations (31.12) for v,

$$v = \frac{1}{b}\left[u' - au\right] \tag{31.17}$$

Substituting (31.17) into both sides of the second of equations (31.12),

$$\frac{1}{b}\left[u'' - au'\right] = v' = cu + dv = cu + \frac{d}{b}\left[u' - au\right] \tag{31.18}$$

Rearranging,

$$u'' - (a + d)u' + (da - bc)u = 0 \tag{31.19}$$

If $b = 0$ and $c \neq 0$ the first of equations (31.12) becomes

$$u' = au \tag{31.20}$$

and the second one can be solved for u,

$$u = \frac{1}{c}\left[v' - dv\right] \tag{31.21}$$

Substituting (31.21) into (31.20),

$$\frac{1}{c}\left[v'' - du'\right] = u' = au = \frac{a}{u}\left[v' - dv\right] \tag{31.22}$$

Rearranging,

$$v'' - (a + d)v' + adv = 0 \tag{31.23}$$

By a similar process we find that if $c \neq 0$

$$v'' - (a + d)v' + (ad - bc)v = 0 \tag{31.24}$$

and that if $c=0$ but $b\neq 0$,

$$u'' - (a+d)u' + adu = 0 \tag{31.25}$$

In every case in which one of the equations is second order, the characteristic equation of the differential equation is identical to the characteristic equation of the matrix, and hence the solutions are linear combinations of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$, if the eigenvalues are distinct, or are linear combinations of $e^{\lambda t}$ and $te^{\lambda t}$ if the eigenvalue is repeated. Furthermore, if one (or both) the equations turn out to be first order, the solution to that equation is still $e^{\lambda t}$ where λ is one of the eigenvalues of $\mathbf{A}$.

Theorem 31.2. Let

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{31.26}$$

Then the solution of $y' = Ay$ is given be one of the following four cases.

1. If $b=c=0$, then the eigenvalues of $\mathbf{A}$ are a and d, and

$$\left.\begin{aligned} y_1 &= C_1 e^{at} \\ y_2 &= C_2 e^{dt} \end{aligned}\right\} \tag{31.27}$$

2. If $b\neq 0$ but $c=0$, then the eigenvalues of $\mathbf{A}$ are a and d, and

$$\left.\begin{aligned} y_1 &= \begin{cases} C_1 e^{at} + C_2 e^{dt} & a\neq d \\ (C_1 + C_2 t)e^{at} & a = d \end{cases} \\ y_2 &= C_3 e^{dt} \end{aligned}\right\} \tag{31.28}$$

3. If $b=0$ but $c\neq 0$, then the eigenvalues of $\mathbf{A}$ are a and d, and

$$\left.\begin{aligned} y_1 &= C_1 e^{dt} \\ y_2 &= \begin{cases} C_2 e^{at} + C_3 e^{dt} & a\neq d \\ (C_2 + C_3 t)e^{at} & a = d \end{cases} \end{aligned}\right\} \tag{31.29}$$

4. If $b\neq 0$ and $c\neq 0$, then the eigenvalues of $\mathbf{A}$ are

$$\lambda = \frac{1}{2}\left[T \pm \sqrt{T^2 - 4\Delta}\right] \tag{31.30}$$

   and the solutions are

   (a) If $\lambda_1 = \lambda_2 = \lambda$

$$y_i = (C_{i1} + C_{i2}t)e^{\lambda t} \tag{31.31}$$

(b) If $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, i.e., $T^2 \geq 4\Delta$,

$$y_i = C_{i1}e^{\lambda_1 t} + C_{i2}e^{\lambda_2 t} \tag{31.32}$$

(c) It $T^2 < 4\Delta$ then $\lambda_{1,2} = \mu \pm i\sigma$, where $\mu, \sigma \in \mathbb{R}$, and

$$y_i = e^{\mu t}(C_{i1}\cos\sigma t + C_{i2}\sin\sigma t) \tag{31.33}$$

Example 31.1. Solve the system

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \quad y(0) = \begin{pmatrix} 6 \\ 5 \end{pmatrix} \tag{31.34}$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \tag{31.35}$$

by elimination of variables.

To eliminate variables means we try to reduce the system to either two separate first order equations or a single second order equation that we can solve. Multiplying out the matrix system gives us

$$u' = u + v, \ u_0 = 6 \tag{31.36}$$

$$v' = 4u - 2v, \ v_0 = 5 \tag{31.37}$$

Solving the first equation for v and substituting into the second,

$$v' = 4u - 2v = 4u - 2(u' - u) = 6u - 2u' \tag{31.38}$$

Differentiating the first equations and substituting

$$u'' = u' + v' = u' + 6u - 2u' = -u' + 6u \tag{31.39}$$

Rearranging,

$$u'' + u' - 6u = 0 \tag{31.40}$$

The characteristic equation is $r^2 + r - 6 = (r-2)(r+3) = 0$ so that

$$u = C_1e^{2t} + C_2e^{-3t} \tag{31.41}$$

From the initial conditions, $u_0' = u_0 + v_0 = 11$. Hence

$$\begin{array}{rcl} C_1 + C_2 & = & 6 \\ 2C_1 - 3C_2 & = & 11 \end{array} \tag{31.42}$$

Multiplying the first of (31.42) by 3 and adding to the second gives $5C_1 = 29$ or $C_1 = 29/5$, and therefore $C_2 = 6 - 29/5 = 1/5$. Thus

$$u = \frac{29}{5}e^{2t} + \frac{1}{5}e^{-3t} \tag{31.43}$$

We can find v by substitution int (31.36):

$$\begin{aligned} v &= u' - u && (31.44)\\ &= \frac{58}{5}e^{2t} - \frac{3}{5}e^{-3t} - \frac{29}{5}e^{2t} - \frac{1}{5}e^{-3t} && (31.45)\\ &= \frac{29}{5}e^{2t} - \frac{4}{5}e^{-3t} \quad \square && (31.46) \end{aligned}$$

The Matrix Exponential

Definition 31.3. Let $\mathbf{M}$ be any square matrix. Then we define the **exponential of the matrix** as

$$\exp(\mathbf{M}) = e^{\mathbf{M}} = I + \mathbf{M} + \frac{1}{2}\mathbf{M}^2 + \frac{1}{3!}\mathbf{M}^3 + \frac{1}{4!}\mathbf{M}^4 + \cdots \tag{31.47}$$

assuming that the series converges.

Theorem 31.4. The solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ with initial conditions $\mathbf{y}(t_0) = \mathbf{y}_0$ is $\mathbf{y} = e^{\mathbf{A}(t-t_0)}\mathbf{y}_0$.

Proof. Use Picard iteration:

$$\mathbf{\Phi}_0(t) = \mathbf{y}_0 \tag{31.48}$$

$$\mathbf{\Phi}_k(t) = \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{A}\mathbf{\Phi}_{k-1}(s)ds \tag{31.49}$$

Then

$$\begin{aligned} \mathbf{\Phi}_1 &= \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{A}\mathbf{\Phi}_0(s)ds && (31.50)\\ &= \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{A}\mathbf{y}_0 ds && (31.51)\\ &= \mathbf{y}_0 + \mathbf{A}\mathbf{y}_0(t-t_0) && (31.52)\\ &= [\mathbf{I} + \mathbf{A}(t-t_0)]\mathbf{y}_0 && (31.53) \end{aligned}$$

and in general

$$\mathbf{\Phi}_k = \left[\mathbf{I} + \sum_{j=1}^{k} \frac{1}{k!}\mathbf{A}^k (t-t_0)^k\right]\mathbf{y}_0 \tag{31.54}$$

We will verify (31.54) by induction. We have already demonstrated it for $k = 1$ (equation 31.53). We take (31.54) as an inductive hypothesis and compute

$$\begin{aligned}
\mathbf{\Phi}_{k+1}(t) &= \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{A}\mathbf{\Phi}_k(s)ds && (31.55)\\
&= \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{A}\left[\mathbf{I} + \sum_{j=1}^{k} \frac{1}{k!}\mathbf{A}^k(s-t_0)^k\right]\mathbf{y}_0 ds && (31.56)\\
&= \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{A}\mathbf{I}\mathbf{y}_0 ds + \int_{t_0}^{t}\sum_{j=1}^{k} \frac{1}{k!}\mathbf{A}^{k+1}(s-t_0)^k\mathbf{y}_0 ds && (31.57)\\
&= \mathbf{y}_0 + \mathbf{A}\mathbf{I}\mathbf{y}_0(t-t_0) + \sum_{j=1}^{k} \frac{1}{k!}\mathbf{A}^{k+1}\int_{t_0}^{t}(s-t_0)^k\mathbf{y}_0 ds && (31.58)\\
&= \mathbf{y}_0 + \mathbf{A}\mathbf{I}\mathbf{y}_0(t-t_0) + \sum_{j=1}^{k} \frac{1}{(k+1)!}\mathbf{A}^{k+1}(t-t_0)^{k+1}\mathbf{y}_0 && (31.59)\\
&= \mathbf{y}_0 + \mathbf{A}\mathbf{I}\mathbf{y}_0(t-t_0) + \sum_{j=2}^{k+1} \frac{1}{j!}\mathbf{A}^{j}(t-t_0)^{j}\mathbf{y}_0 && (31.60)\\
&= \mathbf{y}_0 + \sum_{j=1}^{k+1} \frac{1}{j!}\mathbf{A}^{j}(t-t_0)^{j}\mathbf{y}_0 && (31.61)\\
&= \left[\mathbf{I} + \sum_{j=1}^{k+1} \frac{1}{k!}\mathbf{A}^k(t-t_0)^k\right]\mathbf{y}_0 && (31.62)
\end{aligned}$$

which completes the proof of equation (31.54). The general existence theorem (Picard) then says that

$$\mathbf{y} = \left[\mathbf{I} + \sum_{j=1}^{\infty} \frac{1}{k!}\mathbf{A}^k(t-t_0)^k\right]\mathbf{y}_0 \tag{31.63}$$

If we define a matrix $\mathbf{M} = \mathbf{A}(t - t_0)$ and observe that $M^0 = I$, then

$$\mathbf{y} = \left[\mathbf{M}^0 + \sum_{j=1}^{\infty} \frac{1}{k!}\mathbf{M}^k\right]\mathbf{y}_0 = e^{\mathbf{M}}\mathbf{y}_0 = e^{\mathbf{A}(t-t_0)}\mathbf{y}_0 \tag{31.64}$$

as required. □

Example 31.2. Solve the system

$$\mathbf{y}' = \mathbf{A}\mathbf{y} \text{ where } \mathbf{y}(0) = \begin{pmatrix} 6 \\ 5 \end{pmatrix} \tag{31.65}$$

and

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \tag{31.66}$$

From theorem 31.4

$$\mathbf{y} = e^{\mathbf{A}(t-t_0)}\mathbf{y}_0 = \left\{ \exp\left[\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} t \right] \right\} \begin{pmatrix} 6 \\ 5 \end{pmatrix} \tag{31.67}$$

The problem now is that we don't know how to calculate the matrix exponential! We will continue this example after we have discussed some of its properties. □

Theorem 31.5. Properties of the Matrix Exponential.

1. If $\mathbf{0}$ is a square $n \times n$ matrix composed entirely of zeros, then
$$e^{\mathbf{0}} = \mathbf{I} \tag{31.68}$$

2. If $\mathbf{A}$ and $\mathbf{B}$ are square $n \times n$ matrices that commute (i.e., if $AB = BA$), then
$$e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}} e^{\mathbf{B}} \tag{31.69}$$

3. $e^{\mathbf{A}}$ is invertible, and
$$\left(e^{\mathbf{A}}\right)^{-1} = e^{-\mathbf{A}} \tag{31.70}$$

4. If $\mathbf{A}$ is any $n \times n$ matrix and $\mathbf{S}$ is any non-singular $n \times n$ matrix, then
$$\mathbf{S}^{-1} e^{\mathbf{A}} \mathbf{S} = \exp\left(\mathbf{S}^{-1}\mathbf{A}\mathbf{S}\right) \tag{31.71}$$

5. Let $\mathbf{D} = \text{diag}(x_1, ..., x_n)$ be a diagonal matrix. Then
$$e^{\mathbf{D}} = \text{diag}(e^{x_1}, ..., e^{x_n}). \tag{31.72}$$

6. If $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix, then $e^{\mathbf{A}} = \mathbf{S}e^{\mathbf{D}}\mathbf{S}^{-1}$.

7. If $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{x}_1, ..., \mathbf{x}_n$ with corresponding eigenvalues $\lambda_1, ..., \lambda_n$, then $e^{\mathbf{A}} = \mathbf{U}\mathbf{E}\mathbf{U}^{-1}$, where $\mathbf{U} = (\mathbf{x}_1, ..., \mathbf{x}_n)$ and $\mathbf{E} = \text{diag}\left(e^{\lambda_1}, ..., e^{\lambda_n}\right)$.

8. The matrix exponential is differentiable and $\dfrac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}$.

The Jordan Form

Let $\mathbf{A}$ be a square $n \times n$ matrix.

$\mathbf{A}$ is **diagonalizable** if for some matrix $\mathbf{U}$, $\mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \mathbf{D}$ is diagonal.

$\mathbf{A}$ is diagonalizable if and only if it has n linearly independent eigenvectors $\mathbf{v}_1, ..., \mathbf{v}_n$, in which case $\mathbf{U} = \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{pmatrix}$ will diagonalize $\mathbf{A}$.

If $\lambda_1, ..., \lambda_k$ are the eigenvalues of $\mathbf{A}$ with multiplicities $m_1, ..., m_k$ (hence $n = m_1 + \cdots + m_k$) then the **Jordan Canonical Form** of $\mathbf{A}$ is

$$\mathbf{J} = \begin{pmatrix} \mathbf{B}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{B}_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{B}_k \end{pmatrix} \tag{31.73}$$

where (a) if $m_i = 1$, $\mathbf{B}_i = \lambda_i$ (a scalar or 1×1 matrix; and (b) if $m_i \neq 1$, $\mathbf{B}_i$ is an $m_i \times m_i$ submatrix with λ_i repeated in every diagonal element, the number 1 in the supra-diagonal, and zeroes everywhere else, e.g., if $m_i = 3$ then

$$\mathbf{B}_i = \begin{pmatrix} \lambda_i & 1 & 0 \\ 0 & \lambda_i & 1 \\ 0 & 0 & \lambda_i \end{pmatrix} \tag{31.74}$$

The $\mathbf{B}_i$ are called ***Jordan Blocks***.

For every square matrix $\mathbf{A}$, there exists a matrix $\mathbf{U}$ such $\mathbf{J} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}$ exists.

If $\mathbf{J}$ is the Jordan form of a matrix, then

$$e^{\mathbf{J}} = \begin{pmatrix} e^{\mathbf{B}_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\mathbf{B}_k} \end{pmatrix} \tag{31.75}$$

Thus $e^{\mathbf{A}} = \mathbf{U}e^{\mathbf{J}}\mathbf{U}^{-1}$.

Example 31.3. Evaluate the solution

$$y = e^{A(t-t_0)}y_0 = \left\{ \exp\left[\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} t \right] \right\} \begin{pmatrix} 6 \\ 5 \end{pmatrix} \tag{31.76}$$

found in the previous example.

The eigenvalues of $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$ are $\lambda = 2, -3$, which we find by solving the the characteristic equation. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the eigenvectors of $\mathbf{A}$. From the last theorem we know that

$$e^{\mathbf{A}t} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \tag{31.77}$$

where $\mathbf{x}_1$ is the eigenvector with eigenvalue 2 and $\mathbf{x}_2$ is the eigenvector with eigenvalue -3.

Let $\mathbf{x}_i = \begin{pmatrix} u_i \\ v_i \end{pmatrix}$.

Then since $\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i$,

$$\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = 2 \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \qquad \Longrightarrow u_1 = v_1 \tag{31.78}$$

$$\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = -3 \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \qquad \Longrightarrow v_2 = -4u_2 \tag{31.79}$$

Since one component of each eigenvector is free, we are free to choose the following eigenvectors:

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \mathbf{x_2} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \tag{31.80}$$

Let

$$\mathbf{U} = \begin{pmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \tag{31.81}$$

Then

$$\mathbf{U}^{-1} = \frac{1}{\det \mathbf{U}} [\text{cof}(\mathbf{U})]^T = -\frac{1}{5} \begin{pmatrix} -4 & -1 \\ -1 & 1 \end{pmatrix} \tag{31.82}$$

Therefore

$$e^{\mathbf{A}t} = \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix} \tag{31.83}$$

$$= \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} 4e^{2t} & e^{2t} \\ e^{-3t} & -e^{-3t} \end{pmatrix} \tag{31.84}$$

$$= \frac{1}{5} \begin{pmatrix} 4e^{2t} + e^{-3t} & e^{2t} - e^{-3t} \\ 4e^{2t} - 4e^{-3t} & e^{2t} + 4e^{-3t} \end{pmatrix} \tag{31.85}$$

$$= \frac{1}{5} \left[\begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} e^{-3t} \right] \tag{31.86}$$

Hence the solution is

$$
\begin{aligned}
\mathbf{y} &= \frac{1}{5}\left[\begin{pmatrix}4 & 1\\ 4 & 1\end{pmatrix}e^{2t} + \begin{pmatrix}1 & -1\\ -4 & 4\end{pmatrix}e^{-3t}\right]\begin{pmatrix}6\\5\end{pmatrix} && (31.87)\\
&= \frac{1}{5}\left[\begin{pmatrix}29\\29\end{pmatrix}e^{2t} + \begin{pmatrix}1\\-4\end{pmatrix}e^{-3t}\right] && (31.88)\\
&= \frac{29}{5}e^{2t}\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{5}e^{-3t}\begin{pmatrix}1\\-4\end{pmatrix} && (31.89)
\end{aligned}
$$

We observe in passing that the last line has the form $\mathbf{y} = a\mathbf{x}_1 e^{\lambda_1 t} + b\mathbf{x}_2 e^{\lambda_2 t}$.
□

Properties of Solutions

Theorem 31.6. $\mathbf{y} = e^{\lambda t}\mathbf{v}$ is a solution of the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ if and only if $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ.

Proof. Suppose that $\{\lambda, \mathbf{v}\}$ are an eigenvalue/eigenvector pair for the matrix $\mathbf{A}$. Then

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{31.90}$$

and

$$\frac{d}{dt}\left(e^{\lambda t}\mathbf{v}\right) = \lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}\mathbf{A}\mathbf{v} = \mathbf{A}\left(e^{\lambda t}\mathbf{v}\right) \tag{31.91}$$

Hence $\mathbf{y} = e^{\lambda t}\mathbf{v}$ is a solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

To prove the converse, suppose that $\mathbf{y} = e^{\lambda t}\mathbf{v}$ for some number $\lambda \in \mathbb{C}$ and some vector $\mathbf{v}$, is a solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$. Then $\mathbf{y} = e^{\lambda t}\mathbf{v}$must satisfy the differential equation, so

$$\lambda e^{\lambda t}\mathbf{v} = y' = \mathbf{A}\mathbf{y} = \mathbf{A}e^{\lambda t}\mathbf{v} \tag{31.92}$$

Dividing by the common scalar factor of $e^{\lambda t}$, which can never be zero, gives $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. Hence λ is an eigenvalue of $\mathbf{A}$ with eigenvector $\mathbf{v}$. □

Theorem 31.7. The matrix $e^{\mathbf{A}t}$ is a fundamental matrix of $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

Proof. Let $\mathbf{W} = e^{\mathbf{A}t}$. From the previous theorem,

$$\mathbf{W}' = \frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} = \mathbf{A}\mathbf{W} \tag{31.93}$$

Let $\mathbf{y}_1, ..., \mathbf{y}_n$ denote the column vectors of $\mathbf{W}$. Then by (31.93)

$$\begin{aligned}
\begin{pmatrix}\mathbf{y}_1' & \cdots & \mathbf{y}_n'\end{pmatrix} &= \mathbf{W}' &(31.94)\\
&= \mathbf{A}\mathbf{W} &(31.95)\\
&= \mathbf{A}\begin{pmatrix}\mathbf{y}_1 & \cdots & \mathbf{y}_n\end{pmatrix} &(31.96)\\
&= \begin{pmatrix}\mathbf{A}\mathbf{y}_1 & \cdots & \mathbf{A}\mathbf{y}_n\end{pmatrix} &(31.97)
\end{aligned}$$

Equating columns,

$$\mathbf{y}_i' = \mathbf{A}\mathbf{y}_i, \quad i = 1, \ldots, n \tag{31.98}$$

hence each column of $\mathbf{W}$ is a solution of the differential equation.

Furthermore, by property (3) of of the Matrix Exponential, $\mathbf{W} = e^{\mathbf{A}t}$ is invertible. Since a matrix is invertible if and only if all of its column vectors are linearly independent, this means that the columns of $\mathbf{W}$ form a linearly independent set of solutions to the differential equation. To prove that they are a fundamental set of solutions, suppose that $\mathbf{y}(t)$ is a solution of the initial value problem with $\mathbf{y}(t_0) = \mathbf{y}_0$.

We must show that it is a linear combination of the columns of $\mathbf{W}$. Since the matrix $\mathbf{W}$ is invertible, the numbers $C_1, C_2, ..., C_n$, which are the components of the vector

$$\mathbf{C} = [\mathbf{W}(t_0)]^{-1}\mathbf{y}_0 \tag{31.99}$$

exist. But

$$\begin{aligned}
\mathbf{\Psi} &= \mathbf{W}\mathbf{C} &(31.100)\\
&= \begin{pmatrix}\mathbf{y}_1 & \cdots & \mathbf{y}_n\end{pmatrix}\begin{pmatrix}C_1\\ \vdots \\ C_n\end{pmatrix} &(31.101)\\
&= C_1\mathbf{y}_1 + \cdots + C_n\mathbf{y}_n &(31.102)
\end{aligned}$$

is a solution of the differential equation, and by (31.99), $\mathbf{\Psi}(t_0) = W(t_0)C = y_0$, so that $\mathbf{\Psi}(t)$ also satisfies the initial value problem. By the uniqueness theorem, $\mathbf{y}$ and $\mathbf{\Psi}(t)$ must be identical.

Hence every solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ is a linear combination of the column vectors of $\mathbf{W}$, because any solution can be considered a solution of some initial value problem. Thus the column vectors form a fundamental set of solutions, and hence $\mathbf{W}$ is a fundamental matrix. $\square$

Theorem 31.8. (Abel's Formula.) The Wronskian of $\mathbf{y}' = \mathbf{A}\mathbf{y}$, where $\mathbf{A}$ is a constant matrix, is

$$W(t) = W(t_0)e^{(t-t_0)\mathrm{trace}(\mathbf{A})} \tag{31.103}$$

If $\mathbf{A}$ is a function of t, then

$$W(t) = W(t_0)\exp\int_{t_0}^{t}[\mathrm{trace}(\mathbf{A}(s))]ds \tag{31.104}$$

Proof. Let $\mathbf{W}$ be a fundamental matrix of $\mathbf{y}' = \mathbf{A}\mathbf{y}$. Then by the formula for differentiation of a determinant,

$$\begin{aligned} W'(t) = &\begin{vmatrix} y_{11}' & y_{21}' & \cdots & y_{n1}' \\ y_{12} & y_{22} & & y_{n2} \\ \vdots & & \ddots & \\ y_{13} & y_{2n} & & y_{nn} \end{vmatrix} + \begin{vmatrix} y_{11} & y_{21} & \cdots & y_{n1} \\ y_{12}' & y_{22}' & & y_{n2}' \\ \vdots & & \ddots & \\ y_{13} & y_{2n} & & y_{nn} \end{vmatrix} \\ &+ \cdots + \begin{vmatrix} y_{11} & y_{21} & \cdots & y_{n1} \\ y_{12} & y_{22} & & \\ \vdots & & \ddots & \\ y_{13}' & y_{2n}' & & y_{nn}' \end{vmatrix} \end{aligned} \tag{31.105}$$

But since $\mathbf{W}$ satisfies the differential equation, $\mathbf{W}' = \mathbf{AW}$, so that

$$\mathbf{W}' = \mathbf{AW} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \mathbf{y}_1 & \cdots & \mathbf{y}_n \end{pmatrix} \tag{31.106}$$

$$= \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{y}_1 & \cdots & \mathbf{a}_1 \cdot \mathbf{y}_n \\ \vdots & & \vdots \\ \mathbf{a}_n \cdot \mathbf{y}_1 & \cdots & \mathbf{a}_n \cdot \mathbf{y}_n \end{pmatrix} \tag{31.107}$$

where $\mathbf{a}_i$ is the ith row vector of $\mathbf{A}$, and the $\mathbf{a}_i \cdot \mathbf{y}_j$ represents the vector dot product between the i th row of $\mathbf{A}$ and the jth solution vector $\mathbf{y}_j$.

Comparing the last two results,

$$W'(t) = \begin{vmatrix} \mathbf{a}_1\cdot\mathbf{y}_1 & \mathbf{a}_1\cdot\mathbf{y}_2 & \cdots & \mathbf{a}_1\cdot\mathbf{y}_n \\ y_{12} & y_{22} & & y_{n2} \\ \vdots & & \ddots & \\ y_{13} & y_{2n} & & y_{nn} \end{vmatrix} + \begin{vmatrix} y_{11} & y_{21} & \cdots & y_{n1} \\ \mathbf{a}_2\cdot\mathbf{y}_1 & \mathbf{a}_2\cdot\mathbf{y}_2 & & \mathbf{a}_2\cdot\mathbf{y}_n \\ \vdots & & \ddots & \\ y_{13} & y_{2n} & & y_{nn} \end{vmatrix}$$
$$+\cdots+ \begin{vmatrix} y_{11} & y_{21} & \cdots & y_{n1} \\ y_{12} & y_{22} & & \\ \vdots & & \ddots & \\ \mathbf{a}_n\cdot\mathbf{y}_1 & \mathbf{a}_n\cdot\mathbf{y}_2 & & \mathbf{a}_n\cdot\mathbf{y}_n \end{vmatrix} \tag{31.108}$$

The first determinant is

$$\begin{vmatrix} \mathbf{a}_1\cdot\mathbf{y}_1 & \cdots & \mathbf{a}_1\cdot\mathbf{y}_n \\ y_{12} & & y_{n2} \\ \vdots & & \vdots \\ y_{13} & \cdots & y_{nn} \end{vmatrix} \tag{31.109}$$
$$= \begin{vmatrix} a_{11}y_{11}+\cdots+a_{1n}y_{1n} & a_{11}y_{21}+\cdots+a_{1n}y_{2n} & \cdots & a_{11}y_{n1}+\cdots+a_{1n}y_{nn} \\ y_{12} & y_{22} & & y_{n2} \\ \vdots & & \ddots & \\ y_{13} & y_{2n} & & y_{nn} \end{vmatrix}$$

We can subtract a_{21} times the second row, a_{31} times the third row, ..., a_{n1} times the nth row from the first row without changing the value of the determinant,

$$\begin{vmatrix} \mathbf{a}_1\cdot\mathbf{y}_1 & \mathbf{a}_1\cdot\mathbf{y}_2 & \cdots & \mathbf{a}_1\cdot\mathbf{y}_n \\ y_{12} & y_{22} & & y_{n2} \\ \vdots & & \ddots & \\ y_{13} & y_{2n} & & y_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}y_{11} & a_{11}y_{21} & \cdots & a_{11}y_{n1} \\ y_{12} & y_{22} & & y_{n2} \\ \vdots & & \ddots & \\ y_{13} & y_{2n} & & y_{nn} \end{vmatrix} \tag{31.110}$$

We can factor out a_{11} from every element in the first row,

$$\begin{vmatrix} \mathbf{a}_1\cdot\mathbf{y}_1 & \mathbf{a}_1\cdot\mathbf{y}_2 & \cdots & \mathbf{a}_1\cdot\mathbf{y}_n \\ y_{12} & y_{22} & & y_{n2} \\ \vdots & & \ddots & \\ y_{13} & y_{2n} & & y_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} y_{11} & y_{21} & \cdots & y_{n1} \\ y_{12} & y_{22} & & y_{n2} \\ \vdots & & \ddots & \\ y_{13} & y_{2n} & & y_{nn} \end{vmatrix} = a_{11}W(t) \tag{31.111}$$

By a similar argument, the second determinant is $a_{22}W(t)$, the third one is $a_{33}W(t)$, ..., and the nth one is $a_{nn}W(t)$. Therefore

$$W'(t) = (a_{11} + a_{22} + \cdots + a_{nn})W(t) = (\text{trace}\mathbf{A})W(t) \tag{31.112}$$

Dividing by $W(t)$ and integrating produces the desired result. $\square$

Theorem 31.9. Let $\mathbf{y}_1, ..., \mathbf{y}_n : I \to \mathbb{R}^n$ be solutions of the linear system $\mathbf{y}' = \mathbf{A}\mathbf{y}$, where $\mathbf{A}$ is an $n \times n$ constant matrix, on some interval $I \subset \mathbb{R}$, and let $W(t)$ denote their Wronskian. Then the following are equivalent:

1. $W(t) \neq 0$ for all $t \in I$.
2. For some $t_0 \in I$, $W(t_0) \neq 0$
3. The set of functions $\mathbf{y}_1(t), ..., \mathbf{y}_n(t)$ are linearly independent.
4. The set of functions $\mathbf{y}_1(t), ..., \mathbf{y}_n(t)$ form a fundamental set of solutions to the system of differential equations on I.

Proof. (1) $\Rightarrow$ (2). Suppose $W(t) \neq 0$ for all $t \in I$ (this is (1)). Then pick any $t_0 \in I$. Then $\exists t_0 \in I$ such that $W(t_0) \neq 0$ (this is (2)).

(2) $\Rightarrow$ (1). This follows immediately from Abel's formula.

(1) $\Rightarrow$ (3). Since the Wronskian is nonzero, the fundamental matrix is invertible. But a matrix is invertible if and only if its column vectors are linearly independent.

(3) $\Rightarrow$ (1). Since the column vectors of the fundamental matrix are linearly independent, this means the matrix is invertible, which in turn implies that its determinant is nonzero.

(3) $\Rightarrow$ (4). This was proven in a previous theorem.

(4) $\Rightarrow$ (3). Since the functions form a fundamental set, they must be linearly independent.

Since (4) $\iff$ (3) $\iff$ (1) $\iff$ (2) all four statements are equivalent. $\square$

Theorem 31.10. Let $\mathbf{A}$ be an $n \times n$ matrix with n linearly independent eigenvectors $\mathbf{v}_1, ..., \mathbf{v}_n$ with corresponding eigenvalues $\lambda_1, ..., \lambda_n$. Then

$$\mathbf{y}_1 = \mathbf{v}_1 e^{\lambda_1 t}, \ldots \mathbf{y}_n = \mathbf{v}_n e^{\lambda_n t} \tag{31.113}$$

form a fundamental set of solutions for $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

Proof. We have show previously that each function in equations (31.113) is a solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$. Since the eigenvectors are linearly independent, the solutions are also linearly independent. It follows that the solutions form a fundamental set. $\square$

Generalized Eigenvectors

As a corollary to theorem 31.10, we observe that if $\mathbf{A}$ is diagonalizable (i.e. it has n linearly independent eigenvectors) then the general solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ is

$$\mathbf{y} = \sum_{i=1}^{n} C_i \mathbf{v}_i e^{\lambda_i t} \tag{31.114}$$

If the matrix is not diagonalizable, then to find the fundamental matrix we must first find the Jordan form $\mathbf{J}$ of $\mathbf{A}$, because

$$e^{\mathbf{A}} = \mathbf{U} e^{\mathbf{J}} \mathbf{U}^{-1} \tag{31.115}$$

where $\mathbf{J} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}$ for some matrix $\mathbf{U}$ that we must determine. If $\mathbf{A}$ were diagonalizable, the columns of $\mathbf{U}$ would be the eigenvectors of $\mathbf{A}$. Since the system is not diagonalizable there are at least two eigenvectors that share the same eigenvalue.

Let $\mathbf{J} = \mathbf{U}^{-\mathbf{1}}\mathbf{A}\mathbf{U}$ be the Jordan Canonical Form of $\mathbf{A}$. Then since $\mathbf{A} = \mathbf{U}\mathbf{J}\mathbf{U}^{-\mathbf{1}}$, we can rearrange the system of differential equations $\mathbf{y}' = \mathbf{A}\mathbf{y}$ as

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \mathbf{U}\mathbf{J}\mathbf{U}^{-\mathbf{1}}\mathbf{y} \tag{31.116}$$

Let $\mathbf{z} = \mathbf{U}^{-\mathbf{1}}\mathbf{y}$. Then

$$\mathbf{U}\mathbf{z}' = \mathbf{y}' = \mathbf{U}\mathbf{J}\mathbf{z} \tag{31.117}$$

Multiplying through by $\mathbf{U}^{-1}$ on the left, we arrive at the Jordan form of the differential equation,

$$\mathbf{z}' = \mathbf{J}\mathbf{z} \tag{31.118}$$

where $\mathbf{J}$ is the Jordan Canonical Form of $\mathbf{A}$. Hence $\mathbf{J}$ is block diagonal, with each block corresponding to a single eigenvalue λ_i of multiplicity m_i. Writing

$$\mathbf{z}' = \mathbf{J}\mathbf{z} = \begin{pmatrix} \mathbf{B}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{B}_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mathbf{B}_k \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{pmatrix} \tag{31.119}$$

we can replace (31.118) with a set of systems

$$\mathbf{z}_i' = \mathbf{B}_i\mathbf{z}_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 \\ 0 & \lambda_i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i \end{pmatrix} \mathbf{z}_i \tag{31.120}$$

where $\mathbf{B}_i$ is an $m_i \times m_i$ matrix (m_i is the multiplicity of λ_i). Starting with any block, denote the components of $\mathbf{z}_i$ as $\zeta_1, \zeta_2, ..., \zeta_m$, and (for the time being, at least) omit the index i. Then in component form,

$$\zeta_1' = \lambda\zeta_1 + \zeta_2 \tag{31.121}$$

$$\zeta_2' = \lambda\zeta_2 + \zeta_3 \tag{31.122}$$

$$\vdots$$

$$\zeta_{n-1} = \lambda\zeta_{n-1} + \zeta_n \tag{31.123}$$

$$\zeta_n' = \lambda_i\zeta_n \tag{31.124}$$

We can solve this type of system by “back substitution” – that is, start with the last one, and work backwards. This method works because the matrix is upper-triangular. The result is

$$\zeta_m = a_m e^{\lambda t} \tag{31.125}$$

where $a_1, ..., a_m$ are arbitrary constants. Rearranging,

$$\mathbf{z} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_m \end{pmatrix} = \begin{pmatrix} a_1 + a_2 t + \cdots + a_m \frac{t^{m-1}}{(m-1)!} \\ \vdots \\ a_{m-1} + a_m t \\ a_m \end{pmatrix} e^{\lambda t} \tag{31.126}$$

hence

$$\mathbf{z} = a_1 e^{\lambda t} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + a_2 e^{\lambda t} \begin{pmatrix} t \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + a_m e^{\lambda t} \begin{pmatrix} \frac{t^{m-1}}{(m-1)!} \\ \vdots \\ t \\ 1 \end{pmatrix} \tag{31.127}$$

Denoting the standard basis vectors in $\mathbb{R}^m$ as $\mathbf{e}_1, ..., \mathbf{e}_m$,

$$\mathbf{z} = a_1 e^{\lambda t}\mathbf{e}_1 + a_2 e^{\lambda t}(\mathbf{e}_2 + \mathbf{e}_1 t) + a_3 e^{\lambda t}\left(\mathbf{e}_3 + \mathbf{e}_2 t + \frac{1}{2}\mathbf{e}_1 t^2\right) + \\ + \cdots + a_m e^{\lambda t}\left(\mathbf{e}_m + \mathbf{e}_{m-1}t + \cdots + \mathbf{e}_1 \frac{t^{m-1}}{(m-1)!}\right) \quad (31.128)$$

Definition 31.11. Let $(\lambda, \mathbf{v})$ be an eigenvalue-eigenvector pair of $\mathbf{A}$ with multiplicity m. Then the set of **generalized eigenvectors** of $\mathbf{A}$ corresponding to the eigenvalue λ are the vectors $\mathbf{w}_1, ..., \mathbf{w}_m$ where

$$(\mathbf{A} - \lambda\mathbf{I})^k \mathbf{w}_k = 0, \quad k = 1, 2, ..., m \quad (31.129)$$

For $k = 1$, equation (31.129) gives

$$0 = (\mathbf{A} - \lambda\mathbf{I})\mathbf{w}_1 = \mathbf{A}\mathbf{w}_1 - \lambda\mathbf{w}_1 \quad (31.130)$$

i.e.,

$$\mathbf{w_1} = \mathbf{v} \quad (31.131)$$

So one of the generalized eigenvectors corresponding to the eigenvector $\mathbf{v}$ is always the original eigenvector $\mathbf{v}$ itself. If $m = 1$, this is the only generalized eigenvector. If $m > 1$, there are additional generalized eigenvectors.

For $k = 2$, equation (31.129) gives

$$\begin{aligned} 0 &= (\mathbf{A} - \lambda\mathbf{I})^2 \mathbf{w}_2 && (31.132)\\ &= (\mathbf{A} - \lambda\mathbf{I})(\mathbf{A} - \lambda\mathbf{I})\mathbf{w}_2 && (31.133)\\ &= \mathbf{A}(\mathbf{A} - \lambda\mathbf{I})\mathbf{w}_2 - \lambda(\mathbf{A} - \lambda\mathbf{I})\mathbf{w}_2 && (31.134) \end{aligned}$$

Rearranging,

$$\mathbf{A}(\mathbf{A} - \lambda\mathbf{I})\mathbf{w}_2 = \lambda(\mathbf{A} - \lambda\mathbf{I})\mathbf{w}_2 \quad (31.135)$$

Thus $(\mathbf{A} - \lambda\mathbf{I})\mathbf{w}_2$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ. Since $\mathbf{w}_1 = \mathbf{v}$ is also an eigenvector with eigenvalue λ, we call it a generalized eigenvector. Thus we also have, from equation 31.135 and 31.129

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{w}_2 = \mathbf{w}_1 \quad (31.136)$$

In general, if the multiplicity of the eigenvalue is m,

$$\begin{aligned} (\mathbf{A} - \lambda\mathbf{I})\mathbf{w}_1 &= 0 && (31.137)\\ (\mathbf{A} - \lambda\mathbf{I})\mathbf{w}_2 &= \mathbf{w}_1 && (31.138)\\ &\vdots \\ (\mathbf{A} - \lambda\mathbf{I})\mathbf{w}_m &= \mathbf{w}_{m-1} && (31.139) \end{aligned}$$

Corollary 31.12. The generalized eigenvectors of the $m \times m$ matrix

$$\mathbf{B} = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda \end{pmatrix} \tag{31.140}$$

are the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_m$.

Proof.

$$\begin{aligned}
(\mathbf{B} - \lambda\mathbf{I})\mathbf{e}_1 &= \lambda\mathbf{e}_1 - \lambda\mathbf{e}_1 = 0 && (31.141)\\
(\mathbf{B} - \lambda\mathbf{I})\mathbf{e}_2 &= \mathbf{B}\mathbf{e}_2 - \lambda\mathbf{e}_2 && (31.142)\\
&= (1, \lambda, 0, \dots, 0)^T - (0, \lambda, 0, \dots, 0,)^T = \mathbf{e}_1 && (31.143)\\
(\mathbf{B} - \lambda\mathbf{I})\mathbf{e}_3 &= (0, 1, \lambda, 0\ \dots, 0)^T - (0, 0, \lambda, 0, \dots, 0)^T = \mathbf{e}_2 && (31.144)\\
&\ \vdots \\
(\mathbf{B} - \lambda\mathbf{I})\mathbf{e}_m &= (0, \dots, 0, 1, \lambda))^T - (0, \dots, 0, 1, 0)^T = \mathbf{e}_{m-1} && (31.145)
\end{aligned}$$

□

Returning to equation (31.128), which gave the solution $\mathbf{z}$ of the ith Jordan Block of the Jordan form $\mathbf{z}' = \mathbf{J}\mathbf{z}$ of the differential equation, we will now make the transformation back to the space of the usual $\mathbf{y}$ variable, using the fact that $\mathbf{y} = \mathbf{U}\mathbf{z}$

$$\begin{aligned}
\mathbf{y} &= \mathbf{U}\mathbf{z} && (31.146)\\
&= a_1 e^{\lambda t}\mathbf{w}_1 + a_2 e^{\lambda t}\left(\mathbf{w}_2 + \mathbf{w}_1 t\right) \\
&\quad + \cdots + a_m e^{\lambda t}\left(\mathbf{w}_m + \mathbf{w}_{m-1} t + \cdots + \mathbf{w}_1 \frac{t^{m-1}}{(m-1)!}\right) && (31.147)
\end{aligned}$$

where

$$\mathbf{w}_i = \mathbf{U}\mathbf{e}_i \tag{31.148}$$

are the generalized eigenvectors of $\mathbf{A}$. This establishes the following theorem.

Theorem 31.13. Let $\lambda_1, \lambda_2, ..., \lambda_j$ be the eigenvalues of $\mathbf{A}$ corresponding to distinct, linearly independent, eigenvectors $\mathbf{v}_1, ..., \mathbf{v}_j$. Denote the multiplicity of eigenvalue λ_i by k_i, for $i = 1, 2, ..., j$ (i.e., $\sum k_i = n$). Let $\mathbf{w}_{i1}, \cdots, \mathbf{w}_{ij}$ be the generalized eigenvectors for λ_i, with $\mathbf{w}_{i1} = \mathbf{v}_i$. Then a fundamental set of solutions to $\mathbf{y}' = \mathbf{Ay}$ are

$$\mathbf{y}_{i1} = \mathbf{w}_{i1} e^{\lambda_i t} \tag{31.149}$$

$$\mathbf{y}_{i2} = (t\mathbf{w}_{i1} + \mathbf{w}_{i2}) e^{\lambda_i t} \tag{31.150}$$

$$\mathbf{y}_{i3} = \left(\frac{t^2}{2} \mathbf{w}_{i1} + t\mathbf{w}_{i2} + \mathbf{w}_{i3} \right) e^{\lambda_i t} \tag{31.151}$$

$$\vdots$$

$$\mathbf{y}_{ij} = \sum_{m=1}^{k_i} \frac{t^{k_i - m}}{(k_i - m)!} \mathbf{w}_m e^{\lambda_i t} \tag{31.152}$$

Example 31.4. Solve

$$\mathbf{y}' = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix} \mathbf{y} \tag{31.153}$$

The characteristic polynomial is

$$0 = \begin{vmatrix} 3 - \lambda & 3 \\ 0 & 3 - \lambda \end{vmatrix} = (3 - \lambda)^2 \tag{31.154}$$

which has $\lambda = 3$ as a solution of multiplicity 2. Letting $\begin{pmatrix} a \\ b \end{pmatrix}$ denote the eigenvector,

$$\begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 3 \begin{pmatrix} a \\ b \end{pmatrix} \tag{31.155}$$

which can be decomposed into the following pair of equations

$$\begin{aligned} 3a + 3b &= 3a \\ 3b &= 3b \end{aligned} \tag{31.156}$$

Hence there is only one eigenvector corresponding to $\lambda = 3$, namely (any multiple of) $(1\ 0)^T$. Therefore one solution of the differential equation is

$$\mathbf{y}_1 = \mathbf{v} e^{\lambda t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} \tag{31.157}$$

and a second, linearly independent solution, is

$$\mathbf{y}_2 = (t\mathbf{v} + \mathbf{w}_2) e^{3t} \tag{31.158}$$

where $(\mathbf{A}-\lambda\mathbf{I})\mathbf{w}_2 = \mathbf{w}_1$, i.e.,

$$\left\{\begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}\mathbf{w}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{31.159}$$

Letting $\mathbf{w}_2 = \begin{pmatrix} c \\ d \end{pmatrix}$, this simplifies to

$$\begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{31.160}$$

which gives $d = 1/3$ and leaves c undetermined (namely, any value will do, so we choose zero). Hence $\mathbf{w}_2 = \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}$:

$$\mathbf{y}_2 = e^{3t}\left[t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1/3 \end{pmatrix}\right] = e^{3t}\begin{pmatrix} t \\ 1/3 \end{pmatrix} \tag{31.161}$$

The general solution is then

$$\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 \tag{31.162}$$

$$= c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} + c_2\begin{pmatrix} t \\ 1/3 \end{pmatrix} e^{3t} \tag{31.163}$$

where c_1 and c_2 are arbitrary constants. *qed*

Variation of Parameters for Systems

Theorem 31.14. A particular solution of

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}(t) \tag{31.164}$$

is

$$\mathbf{y}_P = e^{\mathbf{A}t}\int e^{-\mathbf{A}t}\mathbf{g}(t)dt \tag{31.165}$$

Proof. Let $\mathbf{W}$ be a fundamental matrix of the homogeneous equation $\mathbf{y}' = \mathbf{A}\mathbf{y}$, and denote a fundamental set of solutions by $\mathbf{y}_1, ..., \mathbf{y}_n$. We will look for a particular solution of the form

$$\mathbf{y}_P = \sum_{i=1}^{n} \mathbf{y}_i(t)u_i(t) = \mathbf{W}\mathbf{u} \tag{31.166}$$

for some set of unknown functions $u_1(t), ..., u_n(t)$, and $\mathbf{u} = (u_1(t) \;\; u_2(t) \;\; \cdots \;\; u_n(t))^T$. Differentiating (31.166) gives

$$\mathbf{y}_P' = (\mathbf{Wu})' = \mathbf{W}'\mathbf{u} + \mathbf{Wu}' \tag{31.167}$$

Substitution into the differential equation (31.164) gives

$$\mathbf{W}'\mathbf{u} + \mathbf{Wu}' = \mathbf{AWu} + \mathbf{g} \tag{31.168}$$

Since $\mathbf{W}' = \mathbf{AW}$,

$$\mathbf{AWu} + \mathbf{Wu}' = \mathbf{AWu} + \mathbf{g} \tag{31.169}$$

Subtracting the commong term $\mathbf{AWu}$ gives

$$\mathbf{Wu}' = \mathbf{g} \tag{31.170}$$

Multiplying both sides of the equation by $\mathbf{W}^{-1}$ gives

$$\frac{d\mathbf{u}}{dt} = \mathbf{u}' = \mathbf{W}^{-1}\mathbf{g} \tag{31.171}$$

Since $\mathbf{W} = e^{\mathbf{A}t}$, $\mathbf{W}^{-1} = e^{-\mathbf{A}t}$, and therefore

$$\mathbf{u}(t) = \int \frac{d\mathbf{u}}{dt} dt = \int e^{-\mathbf{A}t}\mathbf{g}(t)dt \tag{31.172}$$

Substition of (31.172) into (31.166) yields the desired result, using the fact that $\mathbf{W} = e^{\mathbf{A}t}$. $\square$

Example 31.5. Find a particular solution to the system

$$\left.\begin{aligned} x' &= x + 3y + 5 \\ y' &= 4x - 3y + 6t \end{aligned}\right\} \tag{31.173}$$

The problem can be restated in the form $\mathbf{y}' = \mathbf{Ay} + \mathbf{g}$ where

$$\mathbf{A} = \begin{pmatrix} 1 & 3 \\ 4 & -3 \end{pmatrix}, \quad \text{and } \mathbf{g}(t) = \begin{pmatrix} 5 \\ 6t \end{pmatrix} \tag{31.174}$$

We first solve the homogeneous system. The eigenvalues of $\mathbf{A}$ satisfy

$$0 = (1 - \lambda)(-3 - \lambda) - 12 \tag{31.175}$$
$$= \lambda^2 + 2\lambda - 15 \tag{31.176}$$
$$= (\lambda - 3)(\lambda + 5) \tag{31.177}$$

so that $\lambda = 3, -5$. The eigenvectors satisfy

$$\begin{pmatrix} 1 & 3 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 3 \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow 2a = 3b \Rightarrow \mathbf{v}_3 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{31.178}$$

and

$$\begin{pmatrix} 1 & 3 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = -5 \begin{pmatrix} c \\ d \end{pmatrix} \Rightarrow d = -2c \Rightarrow \mathbf{v}_{-5} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{31.179}$$

Hence

$$\mathbf{y}_H = c_1 e^{3t} \begin{pmatrix} 3 \\ 2 \end{pmatrix} + c_2 e^{-5t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{31.180}$$

where c_1, c_2 are arbitrary constants. Define a matrix $\mathbf{U}$ whose columns are the eigenvectors of $\mathbf{A}$. Then

$$\mathbf{U} = \begin{pmatrix} 3 & 1 \\ 2 & -2 \end{pmatrix} \text{ and } \mathbf{U}^{-1} = \begin{pmatrix} 1/4 & 1/8 \\ 1/4 & -3/8 \end{pmatrix} \tag{31.181}$$

Then

$$e^{\mathbf{A}t} = \mathbf{U} e^{\mathbf{D}t} \mathbf{U}^{-1} \tag{31.182}$$

where $\mathbf{D} = \text{diag}(3,\ -5)$. Hence

$$e^{\mathbf{A}t} = \begin{pmatrix} 3 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-5t} \end{pmatrix} \begin{pmatrix} 1/4 & 1/8 \\ 1/4 & -3/8 \end{pmatrix} \tag{31.183}$$

$$= \begin{pmatrix} 3 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} e^{3t}/4 & e^{3t}/8 \\ e^{-5t}/4 & -3e^{-5t}/8 \end{pmatrix} \tag{31.184}$$

$$= \begin{pmatrix} \frac{3}{4}e^{3t} + \frac{1}{4}e^{-5t} & \frac{3}{8}e^{3t} - \frac{3}{8}e^{-5t} \\ \frac{1}{2}e^{3t} - \frac{1}{2}e^{-5t} & \frac{1}{4}e^{3t} + \frac{3}{4}e^{-5t} \end{pmatrix} \tag{31.185}$$

Since

$$e^{-\mathbf{A}t} = (\mathbf{U} e^{\mathbf{D}t} \mathbf{U}^{-1})^{-1} = \mathbf{U} e^{-\mathbf{D}t} \mathbf{U}^{-1} \tag{31.186}$$

we have

$$e^{-\mathbf{A}t} = \begin{pmatrix} 3 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{5t} \end{pmatrix} \begin{pmatrix} 1/4 & 1/8 \\ 1/4 & -3/8 \end{pmatrix} \tag{31.187}$$

$$= \begin{pmatrix} \frac{3}{4}e^{-3t} + \frac{1}{4}e^{5t} & \frac{3}{8}e^{-3t} - \frac{3}{8}e^{5t} \\ \frac{1}{2}e^{-3t} - \frac{1}{2}e^{5t} & \frac{1}{4}e^{-3t} + \frac{3}{4}e^{5t} \end{pmatrix} \tag{31.188}$$

By the method of variation of parameters a particular solution is then

$$\mathbf{y}_P = e^{\mathbf{A}t}\int e^{\mathbf{A}t}\mathbf{g}(t)dt \tag{31.189}$$

$$= e^{\mathbf{A}t}\int \begin{pmatrix} \frac{3}{4}e^{-3t}+\frac{1}{4}e^{5t} & \frac{3}{8}e^{-3t}-\frac{3}{8}e^{5t} \\ \frac{1}{2}e^{-3t}-\frac{1}{2}e^{5t} & \frac{1}{4}e^{-3t}+\frac{3}{4}e^{5t}\end{pmatrix}\begin{pmatrix}5\\6t\end{pmatrix}dt \tag{31.190}$$

Using the integral formulas $\int e^{at}dt = \frac{1}{a}e^{at}$ and $\int te^{at}dt = \left(t/a - 1/a^2\right)e^{at}$ we find

$$\begin{aligned}\mathbf{y}_P &= e^{At}\begin{pmatrix} \frac{15}{4(-3)}e^{-3t}+\frac{5}{4(5)}e^{5t}+\frac{9}{4}\left(\frac{t}{-3}-\frac{1}{9}\right)e^{-3t}-\frac{9}{4}\left(\frac{t}{5}-\frac{1}{25}\right)e^{5t} \\ \frac{5}{2(-3)}e^{-3t}-\frac{5}{2(5)}e^{5t}+\frac{3}{2}\left(\frac{t}{-3}-\frac{1}{9}\right)e^{-3t}+\frac{9}{2}\left(\frac{t}{5}-\frac{1}{25}\right)e^{5t}\end{pmatrix} \\ &= e^{At}\begin{pmatrix} -\frac{3}{2}e^{-3t}+\frac{17}{50}e^{5t}-\frac{3t}{4}e^{-3t}-\frac{9t}{20}e^{5t} \\ -e^{-3t}-\frac{17}{25}e^{5t}-\frac{t}{2}e^{-3t}+\frac{9t}{10}e^{5t}\end{pmatrix}\end{aligned} \tag{31.191}$$

Substituting equation (31.183) for $e^{\mathbf{A}t}$

$$\begin{aligned}\mathbf{y}_P &= \begin{pmatrix} \frac{3}{4}e^{3t}+\frac{1}{4}e^{-5t} & \frac{3}{8}e^{3t}-\frac{3}{8}e^{-5t} \\ \frac{1}{2}e^{3t}-\frac{1}{2}e^{-5t} & \frac{1}{4}e^{3t}+\frac{3}{4}e^{-5t}\end{pmatrix}\begin{pmatrix} -\frac{3}{2}e^{-3t}+\frac{17}{50}e^{5t}-\frac{3t}{4}e^{-3t}-\frac{9t}{20}e^{5t} \\ -e^{-3t}-\frac{17}{25}e^{5t}-\frac{t}{2}e^{-3t}+\frac{9t}{10}e^{5t}\end{pmatrix} \\ &= \begin{pmatrix} \left(\frac{3}{4}e^{3t}+\frac{1}{4}e^{-5t}\right)\left(-\frac{3}{2}e^{-3t}+\frac{17}{50}e^{5t}-\frac{3t}{4}e^{-3t}-\frac{9t}{20}e^{5t}\right) \\ +\left(\frac{3}{8}e^{3t}-\frac{3}{8}e^{-5t}\right)\left(-e^{-3t}-\frac{17}{25}e^{5t}-\frac{t}{2}e^{-3t}+\frac{9t}{10}e^{5t}\right) \\ \left(\frac{1}{2}e^{3t}-\frac{1}{2}e^{-5t}\right)\left(-\frac{3}{2}e^{-3t}+\frac{17}{50}e^{5t}-\frac{3t}{4}e^{-3t}-\frac{9t}{20}e^{5t}\right) \\ +\left(\frac{1}{4}e^{3t}+\frac{3}{4}e^{-5t}\right)\left(-e^{-3t}-\frac{17}{25}e^{5t}-\frac{t}{2}e^{-3t}+\frac{9t}{10}e^{5t}\right)\end{pmatrix} \\ &= \begin{pmatrix} -\frac{29}{25}-\frac{6}{5}t \\ -\frac{42}{25}+\frac{2}{5}t\end{pmatrix}\end{aligned} \tag{31.192}$$

Hence

$$\mathbf{y} = \mathbf{y}_H + \mathbf{y}_P \tag{31.193}$$

$$= c_1e^{3t}\begin{pmatrix}3\\2\end{pmatrix} + c_2e^{-5t}\begin{pmatrix}1\\-2\end{pmatrix} + \begin{pmatrix} -\frac{29}{25}-\frac{6}{5}t \\ -\frac{42}{25}+\frac{2}{5}t\end{pmatrix} \tag{31.194}$$

In terms of the variables x and y in the original problem,

$$x = 3c_1e^{3t} + c_2e^{-5t} - \frac{29}{25} - \frac{6}{5}t \tag{31.195}$$

$$y = 2c_1e^{3t} - 2c_2e^{-5t} - \frac{42}{25} + \frac{2}{5}t \quad \square \tag{31.196}$$

Non-constant Coefficients

We can solve the general linear system

$$\mathbf{y}' = \mathbf{A}(t)\mathbf{y}(t) + \mathbf{g}(t) \tag{31.197}$$

where the restriction that $\mathbf{A}$ be constant has been removed, in the same manner that we solved it in the scalar case. We expect

$$\mathbf{M}(t) = \exp\left[-\int \mathbf{A}(t)dt\right] \tag{31.198}$$

to be an integrating factor of (31.197); in fact, since

$$\mathbf{M}'(t) = -\mathbf{M}(t)\mathbf{A}(t) \tag{31.199}$$

we can determine

$$\frac{d}{dt}\left(\mathbf{M}(t)\mathbf{y}\right) = \mathbf{M}(t)\mathbf{y}' + \mathbf{M}'(t)\mathbf{y} \tag{31.200}$$

$$= \mathbf{M}(t)\mathbf{y}' - \mathbf{M}(t)\mathbf{A}(t)\mathbf{y} \tag{31.201}$$

$$= \mathbf{M}(t)(\mathbf{y}' - \mathbf{A}(t)\mathbf{y}(t)) \tag{31.202}$$

$$= \mathbf{M}(t)\mathbf{g}(t) \tag{31.203}$$

Hence

$$\mathbf{M}(t)\mathbf{y} = \int \mathbf{M}(t)\mathbf{g}(t)dt + \mathbf{C} \tag{31.204}$$

and therefore the solution of (31.197) is

$$\mathbf{y} = \mathbf{M}^{-1}(t)\left[\int \mathbf{M}(t)\mathbf{g}(t)dt + \mathbf{C}\right] \tag{31.205}$$

for any arbitrary constant vector $\mathbf{C}$.

Example 31.6. Solve $\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{g}(t)$, $\mathbf{y}(0) = \mathbf{y}_0$, where

$$\mathbf{A} = \begin{pmatrix} 0 & -t \\ -t & 0 \end{pmatrix}, \quad \mathbf{g}(t) = \begin{pmatrix} t \\ 3t \end{pmatrix}, \quad \mathbf{y}_0 = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \tag{31.206}$$

To find the integrating factor $\mathbf{M} = \exp\int -\mathbf{A}(t)dt$, we first calculate

$$\mathbf{P}(t) = -\int \mathbf{A}(t)dt = \frac{1}{2}\begin{pmatrix} 0 & t^2 \\ t^2 & 0 \end{pmatrix} \tag{31.207}$$

The eigenvalues of $\mathbf{P}$ are

$$\lambda_1 = -\frac{t^2}{2}, \quad \lambda_2 = \frac{t^2}{2} \tag{31.208}$$

The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{31.209}$$

The diagonalizing matrix is then

$$\mathbf{S} = (\mathbf{v}_1 \;\; \mathbf{v}_2) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \tag{31.210}$$

and its inverse is given by

$$\mathbf{S}^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \tag{31.211}$$

so that

$$\mathbf{D} = \mathbf{S^{-1}PS} = \frac{1}{2}\begin{pmatrix} -t^2 & 0 \\ 0 & t^2 \end{pmatrix} \tag{31.212}$$

is diagonal. Hence $\mathbf{M} = e^{\mathbf{P}} = \mathbf{S}e^{\mathbf{D}}\mathbf{S}^{-1}$, which we calculate as follows.

$$\begin{aligned}
\mathbf{M} &= \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{-t^2/2} & 0 \\ 0 & e^{t^2/2} \end{pmatrix}\tfrac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \\
&= \tfrac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -e^{-t^2/2} & e^{-t^2/2} \\ e^{t^2/2} & e^{t^2/2} \end{pmatrix} \\
&= \tfrac{1}{2}\begin{pmatrix} e^{-t^2/2}+e^{t^2/2} & -e^{-t^2/2}+e^{t^2/2} \\ -e^{-t^2/2}+e^{t^2/2} & e^{-t^2/2}+e^{t^2/2} \end{pmatrix} \\
&= \begin{pmatrix} \cosh(t^2/2) & \sinh(t^2/2) \\ \sinh(t^2/2) & \cosh(t^2/2) \end{pmatrix}
\end{aligned} \tag{31.213}$$

and (using the identity $\cosh^2 x - \sinh^2 x = 1$),

$$\mathbf{M}^{-1} = \begin{pmatrix} \cosh(t^2/2) & -\sinh(t^2/2) \\ -\sinh(t^2/2) & \cosh(t^2/2) \end{pmatrix} \tag{31.214}$$

Furthermore,

$$\mathbf{M}(t)\mathbf{g}(t) = \begin{pmatrix} \cosh(t^2/2) & \sinh(t^2/2) \\ \sinh(t^2/2) & \cosh(t^2/2) \end{pmatrix}\begin{pmatrix} t \\ 3t \end{pmatrix} \tag{31.215}$$

$$= \begin{pmatrix} t\cosh(t^2/2) + 3t\sinh(t^2/2) \\ t\sinh(t^2/2) + 3t\cosh(t^2/2) \end{pmatrix} \tag{31.216}$$

Using the integral formulas

$$\int t\cosh(t^2/2)dt = \sinh(t^2/2) \tag{31.217}$$

$$\int t\sinh(t^2/2)dt = \cosh(t^2/2) \tag{31.218}$$

we find that

$$\int \mathbf{M}(t)\mathbf{g}(t)dt = \begin{pmatrix} 3\cosh(t^2/2) + \sinh(t^2/2) \\ \cosh(t^2/2) + 3\sinh(t^2/2) \end{pmatrix} \tag{31.219}$$

hence

$$\mathbf{M}^{-1}(t) \int \mathbf{M}(t)\mathbf{g}(t)dt \tag{31.220}$$

$$= \begin{pmatrix} \cosh(t^2/2) & -\sinh(t^2/2) \\ -\sinh(t^2/2) & \cosh(t^2/2) \end{pmatrix} \begin{pmatrix} 3\cosh(t^2/2) + \sinh(t^2/2) \\ \cosh(t^2/2) + 3\sinh(t^2/2) \end{pmatrix} \tag{31.221}$$

$$= \begin{pmatrix} 3 \\ 1 \end{pmatrix} \tag{31.222}$$

Since $\mathbf{y} = \mathbf{M}^{-1}(t)\left[\int \mathbf{M}(t)\mathbf{g}(t)dt + \mathbf{C}\right]$, the general solution is

$$\mathbf{y} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + \begin{pmatrix} \cosh(t^2/2) & -\sinh(t^2/2) \\ -\sinh(t^2/2) & \cosh(t^2/2) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \tag{31.223}$$

$$= \begin{pmatrix} 3 + C_1\cosh(t^2/2) - C_2\sinh(t^2/2) \\ 1 - C_1\sinh(t^2/2) + C_2\cosh(t^2/2) \end{pmatrix} \tag{31.224}$$

The initial conditions give

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 + C_1 \\ 1 + C_2 \end{pmatrix} \tag{31.225}$$

Hence $C_1 = C_2 = 1$ and

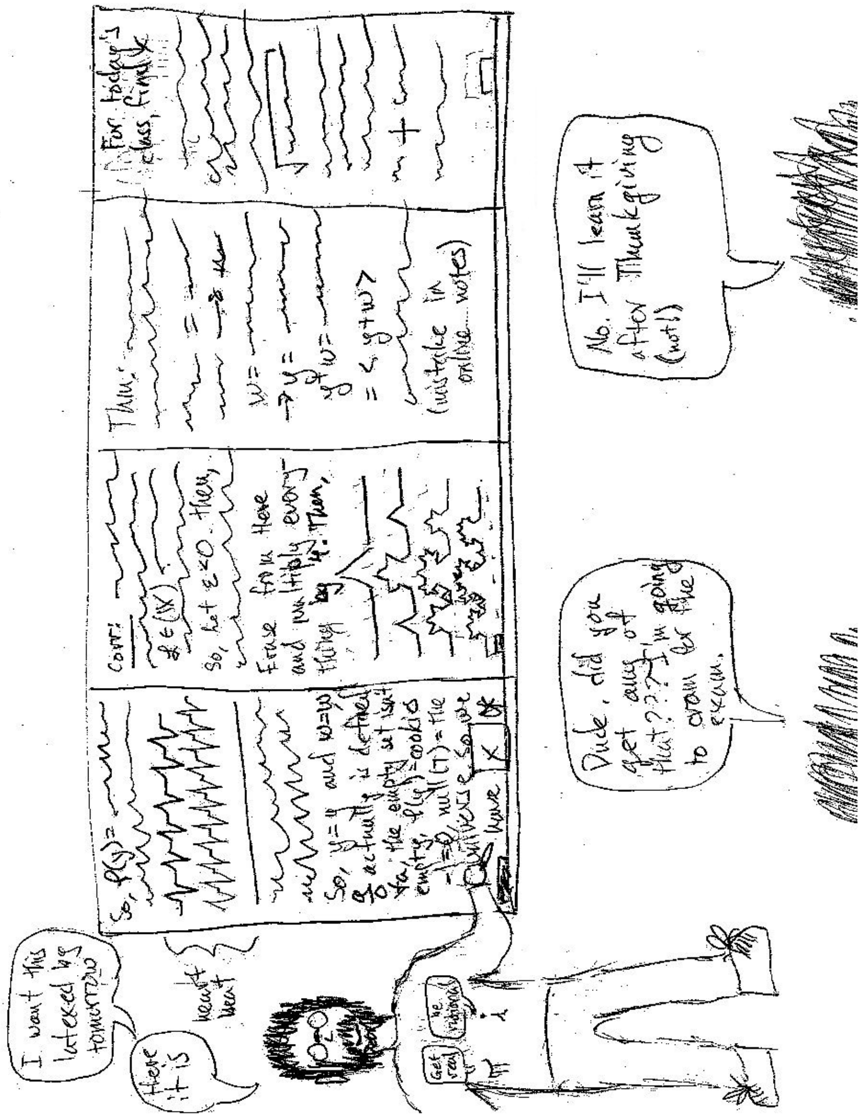
$$\mathbf{y} = \begin{pmatrix} 3 + \cosh(t^2/2) - \sinh(t^2/2) \\ 1 - \sinh(t^2/2) + \cosh(t^2/2) \end{pmatrix} \tag{31.226}$$

Since

$$\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = e^{-x} \tag{31.227}$$

we have the solution of the initial value problem as

$$\mathbf{y} = \begin{pmatrix} 3 + e^{-t^2/2} \\ 1 + e^{-t^2/2} \end{pmatrix}. \quad \square \tag{31.228}$$


I want this latexed by tomorrow
Here it is
Get real
Be rational
Dude, did you get any of that??? I'm going to cram for the exam.
No. I'll learn it after Thanksgiving (not!)

Lesson 32

The Laplace Transform

Basic Concepts

Definition 32.1 (Laplace Transform)**.** We say the **Laplace Transform of the function** $f(t)$ is the function $F(s)$ defined by the integral

$$\boxed{\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)dt} \tag{32.1}$$

provided that integral exists.

The notation $\mathcal{L}[f(t)]$ and $F(s)$ are used interchangeably with one another.

Example 32.1. Find the Laplace Transform of $f(t) = t$.

Solution. From the definition of the Laplace Transform and equation A.53

$$\begin{aligned}
\mathcal{L}[t] &= \int_0^\infty te^{-st}dt && (32.2)\\
&= \left(\frac{t}{s} - \frac{1}{s^2}\right) e^{-st}\bigg|_0^\infty && (32.3)\\
&= \frac{1}{s^2} \quad \square && (32.4)
\end{aligned}$$

Example 32.2. Find the Laplace Transform of $f(t) = e^{2t}$.

Solution. From the definition of the Laplace Transform

$$\mathcal{L}\left[e^{2t}\right] = \int_0^\infty e^{2t}e^{-st}dt = \int_0^\infty e^{(2-s)t}dt \tag{32.5}$$

$$= \frac{1}{2-s}e^{(2-s)t}\Big|_0^\infty \tag{32.6}$$

$$= \frac{1}{s-2} \text{ so long as } s > 2 \quad \square \tag{32.7}$$

Remark. As a generalization of the last example we can observe that

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}, \quad s > a \tag{32.8}$$

Definition 32.2 (Inverse Transform)**.** If $F(s)$ is the Laplace Transform of $f(t)$ then we say that $f(t)$ is the **Inverse Laplace Transform** of $F(s)$ and write

$$f(t) = \mathcal{L}^{-1}[F(s)] \tag{32.9}$$

Example 32.3. From the example 32.1 we can make the following observations:

1. The Laplace Transform of the function $f(t) = t$ is the function $\mathcal{L}[f(t)] = 1/s^2$.
2. The Inverse Laplace Transform of the function $F(s) = 1/s^2$ is the function $f(t) = t$.

In order to prove a condition that will guarantee the existence of a Laplace transform we will need some results from Calculus.

Definition 32.3 (Exponentially Bounded)**.** Suppose that there exist some constants $K > 0$, a, and $M > 0$, such that

$$|f(t)| \le Ke^{at} \tag{32.10}$$

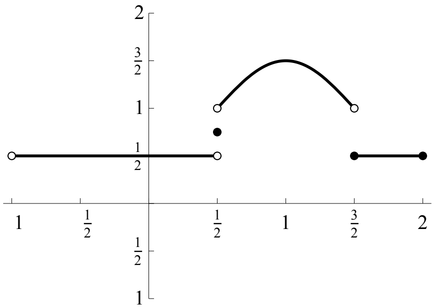
for all $t \ge M$. Then $f(t)$ is said to be **Exponentially Bounded.**

Definition 32.4 (Piecewise Continuous)**.** A function is said to be **Piecewise Continuous** on an interval (a, b) if the interval can be partitioned into a finite number of subintervals

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b \tag{32.11}$$

such that $f(t)$ is continuous on each sub-interval (t_i, t_{i+1}) (figure 32).

Figure 32.1: A piecewise continuous function. This function is piecewise continuous in three intervals $[-1, 1/2], [1/2, 3/2], [3/2, 2]$ and the function approaches a finite limit at the endpoint of each interval as the endpoint is approached from within that interval.



Theorem 32.5 (Integrability)**.** If $f(t)$ is piecewise continuous on (a, b) and $f(t)$ approaches a finite limit at the endpoint of each interval as it is approached from within the interval then $\int_a^b f(t)dt$ exists, i.e., the function is **integrable on (a,b).**

Theorem 32.6 (Existence of Laplace Transform)**.** Let $f(t)$ be defined for all $t \geq 0$ and suppose that $f(t)$ satisfies the following conditions:

1. For any positive constant A, $f(t)$ is piecewise continuous on $[0, A]$.
2. $f(t)$ is **Exponentially Bounded.**

Then the Laplace Transform of $f(t)$ exists for all $s > a$.

Proof. Under the stated conditions, f is piecewise continuous, hence integrable; and $|f(t)| \leq Ke^{at}$ for some K, a, M for all $t \geq M$. Thus

$$\mathcal{L}[F(t)] = \int_0^\infty e^{-st} f(t)dt \tag{32.12}$$

$$= \int_0^M e^{-st} f(t)dt + \int_M^\infty e^{-st} f(t)dt \tag{32.13}$$

Since f is piecewise continuous, so is $e^{-st} f(t)$, and since the definite integral of any piecewise continuous function over a finite domain exists (it is the area under the curves) the first integral exists.

The existence of the Laplace Transform therefore depends on the existence of the second integral We must show that this integral converges. But

$$|e^{-st}f(t)| \leq Ke^{-st}e^{at} = Ke^{(a-s)t} \tag{32.14}$$

Thus the second integral is bounded by

$$\begin{aligned}
\int_M^\infty e^{-st}f(t)dt &\leq \int_M^\infty |e^{-st}f(t)|dt \\
&\leq \int_M^\infty Ke^{(a-s)t}dt \\
&= \lim_{T\to\infty} \int_M^T Ke^{(a-s)t}dt \\
&= \lim_{T\to\infty} \frac{K}{a-s}e^{(a-s)t}\bigg|_M^T \\
&= \lim_{T\to\infty} \frac{K}{a-s}[e^{(a-s)T} - e^{(a-s)M}] \\
&= \begin{cases} 0 & \text{if } a < s \\ \infty & \text{if } a \geq s \end{cases}
\end{aligned}$$

Thus when $s > a$, the second integral in (32.13) vanishes and the total integral converges (is defined). □

Theorem 32.7 (Linearity)**.** The Laplace Transform is lineary, e.g., for any two functions $f(t)$ and $g(t)$ with Laplace Transforms $F(s)$ and $G(s)$, and any two constants A and B,

$$\mathcal{L}[Af(t) + Ag(t)] = A\mathcal{L}[f(t)] + B\mathcal{L}[g(t)] = AF(s) + BG(s) \tag{32.15}$$

Proof. This property follows immediately from the linearity of the integral, as follows:

$$\begin{aligned}
\mathcal{L}[Af(t) + Ag(t)] &= \int_0^\infty (Af(t) + Bg(t))e^{-st}dt & (32.16) \\
&= A\int_0^\infty f(t)e^{-st}dt + B\int_0^\infty g(t)e^{-st}dt & (32.17) \\
&= A\mathcal{L}[f(t)] + B\mathcal{L}[g(t)] = AF(s) + BG(s) & (32.18)
\end{aligned}$$

□

Example 32.4. From integral A.107,

$$\begin{aligned}\mathcal{L}[\cos kt] &= \int_0^\infty \cos kt\, e^{-st}dt \\ &= \left.\frac{e^{-st}(k\sin kt - s\cos kt)}{k^2+s^2}\right|_0^\infty \\ &= \frac{s}{s^2+k^2} \qquad \square\end{aligned}$$

Example 32.5. From integral A.105,

$$\begin{aligned}\mathcal{L}[\sin kt] &= \int_0^\infty \sin kt\, e^{-st}dt \\ &= \left.\frac{e^{-st}(-s\sin kt - k\cos kt)}{k^2+s^2}\right|_0^\infty \\ &= \frac{k}{s^2+k^2} \qquad \square\end{aligned}$$

Example 32.6.

$$\mathcal{L}\left[e^{at}\right] = \int_0^\infty e^{at}e^{-st}dt = \left.\frac{e^{(a-s)t}}{a-s}\right|_0^\infty = \frac{1}{s-a},\ s>a \qquad \square \tag{32.19}$$

Example 32.7. Using linearity and example 32.6,

$$\mathcal{L}[\cosh at]dt = \mathcal{L}\left[\frac{1}{2}(e^{at}+e^{-at})\right] \tag{32.20}$$

$$= \frac{1}{2}\left(\mathcal{L}\left[e^{at}\right] + \mathcal{L}\left[e^{-at}\right]\right) \tag{32.21}$$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) \tag{32.22}$$

$$= \frac{s}{s^2-a^2} \qquad \square \tag{32.23}$$

Example 32.8. Using linearity and example 32.6,

$$\mathcal{L}[\sinh at]dt = \mathcal{L}\left[\frac{1}{2}(e^{at}-e^{-at})\right] \tag{32.24}$$

$$= \frac{1}{2}\left(\mathcal{L}\left[e^{at}\right] - \mathcal{L}\left[e^{-at}\right]\right) \tag{32.25}$$

$$= \frac{1}{2}\left(\frac{1}{s-a} - \frac{1}{s+a}\right) \tag{32.26}$$

$$= \frac{a}{s^2-a^2} \qquad \square \tag{32.27}$$

Example 32.9. Find the Laplace Transform of the step function

$$f(t) = \begin{cases} 0, & 0 \le t \le 3 \\ 2, & t \ge 3 \end{cases} \tag{32.28}$$

Using the definition of $\mathcal{L}[f(t)]$, we compute

$$\mathcal{L}[f(t)] = \int_2^\infty 3e^{-ts}dt = -\frac{3}{s}e^{-ts}\Big|_2^\infty = \frac{3e^{-2s}}{s} \quad \square \tag{32.29}$$

Laplace Transforms of Derivatives

The Laplace Transform of a derivative is what makes it useful in our study of differential equations. Let $f(t)$ be any function with Laplace Transform $F(s)$. Then

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = \int_0^\infty \frac{df(t)}{dt}e^{-st}dt \tag{32.30}$$

We can use integration by parts (which means we use equation A.3) to solve this problem; it is already written in the form $\int udv$ with

$$u = e^{-st} \tag{32.31}$$

$$dvf'(t)dt \tag{32.32}$$

Since this gives $v = f$ and $du = -se^{-st}dt$, we obtain

$$\mathcal{L}[f'(t)] = e^{-st}f(t)\big|_0^\infty + s\int_0^\infty f(t)e^{st}dt \tag{32.33}$$

$$= -f(0) + sF(s) \tag{32.34}$$

which we will write as

$$\boxed{\mathcal{L}[f'(t)] = sF(s) - f(0)} \tag{32.35}$$

Thus **the Laplace Transform of a Derivative is an algebraic function of the Laplace Transform**. That means that Differential equations can be converted into algebraic equations in their Laplace representation, as we will see shortly.

Since the second derivative is the derivative of the first derivative, we can

apply this result iteratively. For example,

$$\mathcal{L}[f''(t)] = \mathcal{L}\left[\frac{d}{dt}f'(t)\right] \tag{32.36}$$
$$= -f'(0) + s\mathcal{L}[f'(t)] \tag{32.37}$$
$$= -f'(0) + s(-f(0) + sF(s)) \tag{32.38}$$
$$= -f'(0) - sf(0) + s^2F(s) \tag{32.39}$$

Theorem 32.8. Let $f, f', \dots, f^{(n-1)}$ be continuous on $[0, \infty)$, of exponential order, and suppose that $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$. Then

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0) \tag{32.40}$$

where $F(s) = \mathcal{L}[f(t)]$.

Proof. This can be proven by mathematical induction. We have already proven the cases for $n = 1$ and $n = 2$. As an inductive hypothesis we will assume (32.40) is true for general $n \geq 1$. We must prove that

$$\mathcal{L}\left[f^{(n+1)}(t)\right] = s^{n+1}F(s) - s^nf(0) - s^{n-1}f'(0) - \cdots - f^{(n)}(0) \tag{32.41}$$

follows directly from (32.40).

Let $g(t) = f^{(n)}$. Then $g'(t) = f^{(n+1)}(t)$ by definition of derivative notation. Hence

$$\mathcal{L}\left[f^{(n+1)}(t)\right] = \mathcal{L}[g'(t)] \tag{32.42}$$
$$= sG(s) - g(0) \tag{32.43}$$
$$= s\mathcal{L}\left[f^{(n)}(t)\right] - f^{(n+1)}(0) \tag{32.44}$$

Substitution of (32.40) yields (32.41) as required. □

The Gamma Function

Definition 32.9. The **Gamma Function** is defined by

$$\boxed{\Gamma(x) = \int_0^\infty u^{x-1}e^{-u}du} \tag{32.45}$$

for $x \in \mathbb{R}$ but x not equal to a negative integer.

Figure 32.2: The Gamma function $\Gamma(t)$.

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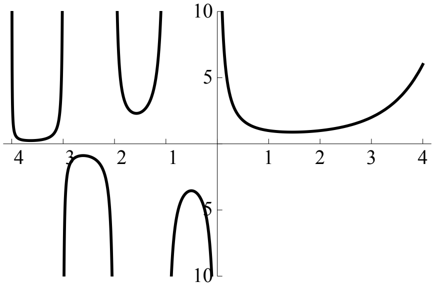


If we make a change of variable $u = st$ for some constant s and new variable t in (32.57), then $du = sdt$ and

$$\Gamma(x) = \int_0^\infty (st)^{x-1}e^{-st}\, sdt = s^x \int_0^\infty t^{x-1}e^{-st}\, dt \tag{32.46}$$

Thus

$$\Gamma(x+1) = s^{x+1}\int_0^\infty t^x e^{-st}\, dt = \mathcal{L}[t^x] \tag{32.47}$$

or

$$\boxed{\mathcal{L}[t^x] = \frac{\Gamma(x+1)}{s^{x+1}}} \tag{32.48}$$

The gamma function has an interesting factorial like property. Decreasing the argument in (32.49) by 1,

$$\Gamma(x) = \mathcal{L}\left[t^{x-1}\right] \tag{32.49}$$

But since

$$t^{x-1} = \frac{1}{x}\frac{d}{dt}t^x \tag{32.50}$$

$$\mathcal{L}\left[t^{x-1}\right] = \mathcal{L}\left[\frac{1}{x}\frac{d}{dt}t^x\right] = \frac{1}{x}\mathcal{L}\left[\frac{d}{dt}t^x\right] \tag{32.51}$$

Let $f(t) = t^x$. Then

$$\mathcal{L}\left[\frac{d}{dt}t^x\right] = \mathcal{L}[f'(t)] = sF(s) - f(0) = sF(s) \tag{32.52}$$

since $f(0) = 0$, where $F(s)$ is given by the right hand side of (32.57) Substituting (32.52) into (32.51) gives

$$\mathcal{L}\left[t^{x-1}\right] = \frac{1}{x}\mathcal{L}\left[\frac{d}{dt}t^x\right] = \frac{s}{x}\frac{\Gamma(x+1)}{s^{x+1}} \tag{32.53}$$

But from (32.51) directly

$$\mathcal{L}\left[t^{x-1}\right] = \frac{\Gamma(x)}{s^x} \tag{32.54}$$

Equating the last two expressions gives

$$\frac{\Gamma(x)}{s^x} = \frac{s}{x}\frac{\Gamma(x+1)}{s^{x+1}} \tag{32.55}$$

which after cancellation gives us the fundamental property of the Gamma function:

$$\boxed{\Gamma(x+1) = x\Gamma(x)} \tag{32.56}$$

Now observe from (32.57) that

$$\Gamma(1) = \int_0^\infty u^{1-1}e^{-u}du = \int_0^\infty e^{-u}du = 1 \tag{32.57}$$

From (32.56)

$$\Gamma(2) = 1 \cdot \Gamma(1) = 1 \cdot 1 = 1 = 1! \tag{32.58}$$
$$\Gamma(3) = 2 \cdot \Gamma(2) = 2 \cdot 1 = 2 = 2! \tag{32.59}$$
$$\Gamma(4) = 3 \cdot \Gamma(3) = 3 \cdot 2 = 6 = 3! \tag{32.60}$$
$$\Gamma(5) = 4 \cdot \Gamma(4) = 4 \cdot 6 = 24 = 4! \tag{32.61}$$
$$\Gamma(6) = 5 \cdot \Gamma(5) = 5 \cdot 4! = 5! \tag{32.62}$$
$$\vdots \tag{32.63}$$
$$\Gamma(n) = (n-1)! \quad \text{for } n \in \mathbb{Z}^+ \tag{32.64}$$

hence for n an integer

$$\boxed{\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}} \tag{32.65}$$

Using Laplace Transforms to Solve Linear Differential Equations

The idea is this: the Laplace Transform turns derivatives into algebraic functions. So if we convert an entire linear ODE into its Laplace Transform, since the transform is linear, all of the derivatives will go away. Then we can solve for $F(s)$ as a function of s. A solution to the differential equation is given by any function $f(t)$ whose Laplace transform is given by $F(s)$. The process is illustrated with an example.

Example 32.10. Solve $y' + 2y = 3$, $y(0) = 1$, using Laplace transforms.

Let $Y(s) = \mathcal{L}[y(t)]$ and apply the Laplace Transform operator to the entire equation.

$$y'(t) + 2y(t) = 3t \tag{32.66}$$

$$\mathcal{L}[y'(t) + 2y(t)] = \mathcal{L}[3t] \tag{32.67}$$

$$\mathcal{L}[y'(t)] + 2\mathcal{L}[y(t)] = 3\mathcal{L}[t] \tag{32.68}$$

From example 32.1, and equation 32.35,

$$sY(s) - y(0) + 2Y(s) = \frac{3}{s^2} \tag{32.69}$$

$$(s+2)Y(s) = \frac{3}{s^2} + y(0) = \frac{3}{s^2} + 1 = \frac{3+s^2}{s^2} \tag{32.70}$$

$$\tag{32.71}$$

Solving for $Y(s)$,

$$Y(s) = \frac{3+s^2}{(s+2)s^2} \tag{32.72}$$

The question then becomes the following: What function has a Laplace Transform that is equal to $(3+s^2)/(s^2(s+2))$? This is called the inverse Laplace Transform of $Y(s)$ and gives $y(t)$:

$$y(t) = \mathcal{L}^{-1}\left[\frac{3+s^2}{(s+2)s^2}\right] \tag{32.73}$$

We typically approach this by trying to simplify the function into smaller parts until we recognize each part as a Laplace transform. For example, we can use the method of Partial Fractions to separated the expression:

$$\frac{3+s^2}{(s+2)s^2} = \frac{A+Bs}{s^2} + \frac{C+Ds}{s+2} \tag{32.74}$$

$$= \frac{(A+Bs)(s+2)}{s^2(s+2)} + \frac{(C+Ds)s^2}{s^2(s+2)} \tag{32.75}$$

Equating numerators and expanding the right hand side,

$$\begin{aligned}
3 + s^2 &= (A + Bs)(s + 2) + (C + Ds)s^2 && (32.76)\\
&= As + 2A + Bs^2 + 2Bs + Cs^2 + Ds^3 && (32.77)\\
&= 2A + (A + 2B)s + (B + C)s^2 + Ds^3 && (32.78)
\end{aligned}$$

Equating coefficients of like powers of s,

$$\begin{aligned}
3 &= 2A && \implies A = \frac{3}{2} && (32.79)\\
0 &= A + 2B && \implies B = \frac{-A}{2} = -\frac{3}{4} && (32.80)\\
1 &= B + C && \implies C = 1 - B = \frac{7}{4} && (32.81)\\
0 &= D && && (32.82)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{3 + s^2}{(s + 2)s^2} &= \frac{\frac{3}{2} - \frac{3}{4}s}{s^2} + \frac{\frac{7}{4}}{s + 2} && (32.83)\\
&= \frac{3}{2} \cdot \frac{1}{s^2} - \frac{3}{4} \cdot \frac{1}{s} + \frac{7}{4} \cdot \frac{1}{s + 2} && (32.84)
\end{aligned}$$

From equations (32.4) and (32.8)

we see that

$$\begin{aligned}
\frac{3 + s^2}{(s + 2)s^2} &= \frac{3}{2} \cdot \mathcal{L}[t] - \frac{3}{4} \cdot \mathcal{L}\left[e^0\right] + \frac{7}{4} \cdot \mathcal{L}\left[e^{-2t}\right] && (32.85)\\
&= \mathcal{L}\left[\frac{3}{2}t - \frac{3}{4} + \frac{7}{4} \cdot e^{-2t}\right] && (32.86)\\
y(t) =& \mathcal{L}^{-1}\left[\frac{3 + s^2}{(s + 2)s^2}\right] = \frac{3}{2}t - \frac{3}{4} + \frac{7}{4} \cdot e^{-2t} && (32.87)
\end{aligned}$$

which gives us the solution to the initial value problem. □

Here is a summary of the procedure to solve the initial value problem for $y'(t)$:

1. Apply the Laplace transform operator to both sides of the differential equation.

2. Evaluate all of the Laplace transforms, including any initial conditions, so that the resulting equation has all $y(t)$ removed and replaced with $Y(s)$.

3. Solve the resulting algebraic equation for $Y(s)$.

4. Find a function $y(t)$ whose Laplace transform is $Y(s)$. This is the solution to the initial value problem.

The reason why this works is because of the following theorem, which we will accept without proof.

Theorem 32.10 (Lerch's Theorem[1]). For any function $F(s)$, there is at most one continuous function function $f(t)$ defined for $t \geq 0$ for which $\mathcal{L}[f(t)] = F(s)$.

This means that there is no ambiguity in finding the inverse Laplace transform.

The method also works for higher order equations, as in the following example.

Example 32.11. Solve $y'' - 7y' + 12y = 0$, $y(0) = 1$, $y'(0) = 1$ using Laplace transforms.

Following the procedure outlined above:

$$\begin{aligned}
0 &= \mathcal{L}[y'' - 7y' + 12y] && (32.88)\\
&= \mathcal{L}[y''] - 7\mathcal{L}[y'] + 12\mathcal{L}[y] && (32.89)\\
&= s^2Y(s) - sy(0) - y'(0) - 7(sY(s) - y(0)) + 12Y(s) && (32.90)\\
&= s^2Y(s) - s - 1 - 7sY(s) + 7 + 12Y(s) && (32.91)\\
&= (s^2 - 7s + 12)Y(s) + 6 - s && (32.92)\\
Y(s) &= \frac{s-6}{s^2 - 7s + 12} = \frac{s-6}{(s-3)(s-4)} = \frac{3}{s-3} - \frac{2}{s-4} && (32.93)
\end{aligned}$$

where the last step is obtained by the method of partial fractions. Hence

$$y(t) = \mathcal{L}^{-1}\left[\frac{3}{s-3} - \frac{2}{s-4}\right] = 3e^{3t} - 2e^{4t}. \quad \square \tag{32.94}$$

[1]Named for Mathias Lerch (1860-1922), an eminent Czech Mathematician who published more than 250 papers.

Derivatives of the Laplace Transform

Now let us see what happens when we differentiate the Laplace Transform. Let $F(s)$ be the transform of $f(t)$.

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt \tag{32.95}$$

$$= \int_0^\infty \frac{d}{ds}e^{-st}f(t)dt \tag{32.96}$$

$$= -\int_0^\infty e^{-st}tf(t)dt \tag{32.97}$$

$$= -\mathcal{L}[tf(t)] \tag{32.98}$$

By a similar argument, we get additional factors of $-t$ for each order of derivative, so in general we have

$$F^{(n)}(s) = \frac{d^{(n)}}{ds^{(n)}}\mathcal{L}[f(t)] = (-1)^n\mathcal{L}[t^n f(t)] \tag{32.99}$$

Thus we have the results

$$\mathcal{L}[tf(t)] = F'(s) \tag{32.100}$$

$$\mathcal{L}[t^n f(t)] = (-1)^n F(n)(s) \tag{32.101}$$

Example 32.12. Find $F(s)$ for $f(t) = t\cos kt$.

From (32.19) we have $\mathcal{L}[\cos kt] = s/(s^2+k^2)$. Hence

$$\mathcal{L}[t\cos kt] = \frac{d}{ds}\mathcal{L}[\cos kt] \tag{32.102}$$

$$= -\frac{d}{ds}\frac{s}{s^2+k^2} \tag{32.103}$$

$$= -\frac{(s^2+k^2)(1) - s(2s)}{(s^2+k^2)^2} \tag{32.104}$$

$$= \frac{s^2-k^2}{(s^2+k^2)^2} \quad \square \tag{32.105}$$

Step Functions and Translations in the Time Variable

Step functions are special cases of piecewise continuous functions, where the function "steps" between two different constant values like the treads on

a stairway. It is possible to define more complicated piecewise continuous functions in terms of step functions.

Definition 32.11. The **Unit Step Function with step at origin**, $\mathcal{U}(t)$ is defined by

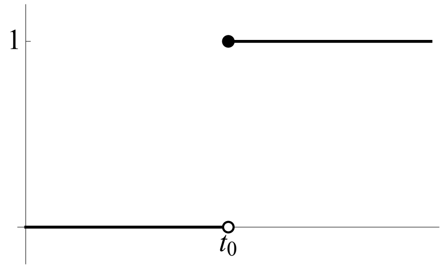
$$\mathcal{U}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases} \tag{32.106}$$

The **Unit Step Function with Step at** t_0, $\mathcal{U}(t - t_0)$ is then given by

$$\mathcal{U}(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t \geq t_0 \end{cases} \tag{32.107}$$

The second function is obtained from the first by recalling that subtraction of t_0 from the argument of a function translates it to the right by t_0 units along the x-axis.

Figure 32.3: A Unit Step Function $\mathcal{U}(t - t_0)$ with step at $t = t_0$.



Unit step functions make it convenient for us to define stepwise continuous functions as simple formulas with the need to list separate cases. For example, the function

$$f(t) = \begin{cases} 0, & t < 1 \\ 1, & 0 \leq t < 2 \\ 0, & t \geq 2 \end{cases} \tag{32.108}$$

can also be written as (figure 32.3)

$$f(t) = \mathcal{U}(t-1) - \mathcal{U}(t-2) \tag{32.109}$$

while the function

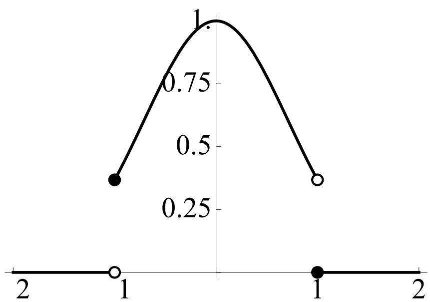
$$f(t) = \begin{cases} 0 & t < -1 \\ e^{-t^2} & -1 \leq t < 1 \\ 0 & t > 1 \end{cases} \tag{32.110}$$

can be written as

$$f(t) = e^{-t^2}(\mathcal{U}(t+1) - \mathcal{U}(t-1) \tag{32.111}$$

as illustrated in figure 32.4.

Figure 32.4: A piecewise continuous function defined with step functions.

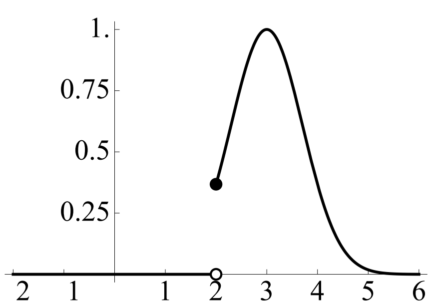


We can also use the combination to produce piecewise continuous translations, for example,

$$f(t) = \begin{cases} e^{-(x-3)^2}, & x \geq 2 \\ 0, & x < 2 \end{cases} = \mathcal{U}(t-2)f(t-3) \tag{32.112}$$

The second factor ($f(t-3)$) translates the bell curve to the right by 3 units, while the first factor cuts off the part to the left of $t = 2$ (figure 32.5).

Figure 32.5: A piecewise continuous function defined by a translation multiplied by a step functions.



The Laplace Transform of the step function is easily calculated,

$$\mathcal{L}[\mathcal{U}(t-a)] = \int_0^\infty \mathcal{U}(t-a)e^{-st}dt \tag{32.113}$$

$$= \int_a^\infty e^{-st}dt \tag{32.114}$$

$$= \frac{e^{-as}}{s} \tag{32.115}$$

We will also find the Laplace Transforms of shifted functions to be useful:

$$\mathcal{L}[f(t)\mathcal{U}(t-a)] = \int_0^\infty f(t)\mathcal{U}(t-a)e^{-st}dt \tag{32.116}$$

$$= \int_a^\infty f(t)e^{-st}dt \tag{32.117}$$

Here we make a change of variable

$$x = t - a \tag{32.118}$$

so that

$$\mathcal{L}[f(t)\mathcal{U}(t-a)] = \int_0^\infty f(x+a)e^{-s(x+a)}dx \tag{32.119}$$

$$= e^{-sa}\int_0^\infty f(x+a)e^{-sx}dx \tag{32.120}$$

$$= e^{-sa}\mathcal{L}[f(t+a)] \tag{32.121}$$

A similar result is obtained for

$$\mathcal{L}[f(t-a)\mathcal{U}(t-a)] = \int_0^\infty f(t-a)\mathcal{U}(t-a)e^{-st}dt \tag{32.122}$$

$$= \int_a^\infty f(t-a)e^{-st}dt \tag{32.123}$$

Now define

$$x = t - a \tag{32.124}$$

so that

$$\mathcal{L}[f(t-a)\mathcal{U}(t-a)] = \int_0^\infty f(t)e^{-s(x+a)}dx \tag{32.125}$$

$$= e^{-sa}\int_0^\infty f(t)e^{-sx}dx \tag{32.126}$$

$$= e^{-as}F(s) \tag{32.127}$$

Translations in the Laplace Variable

If $F(s)$ is the Laplace Transform of $f(t)$,

$$F(s) = \int_0^\infty f(t)e^{-st}dt \tag{32.128}$$

What happens when we shift the s-variable a distance a? Substituting $S = s - a$,

$$\begin{aligned} F(s-a) = F(S) &= \int_0^\infty f(t)e^{-St}dt && (32.129)\\ &= \int_0^\infty f(t)e^{-(s-a)t}dt && (32.130)\\ &= \int_0^\infty e^{at}f(t)e^{-st}dt && (32.131)\\ &= \mathcal{L}\left[e^{at}f(t)\right] && (32.132)\end{aligned}$$

which is often more useful in its inverse form,

$$\mathcal{L}^{-1}[F(s-a)] = e^{at}f(t) \tag{32.133}$$

Example 32.13. Find $f(t)$ such that $F(s) = \dfrac{s+2}{(s-2)^2}$.

Using partial fractions,

$$\begin{aligned} \frac{s+2}{(s-2)^2} &= \frac{A}{s-2} + \frac{B}{(s-2)^2} && (32.134)\\ &= \frac{A(s-2)}{(s-2)^2} + \frac{B}{(s-2)^2} && (32.135)\\ s+2 &= As + (B-2A) && (32.136)\end{aligned}$$

Hence

$$\begin{aligned} A &= 1 && (32.137)\\ B - 2A = 2 &\implies B = 4 && (32.138)\end{aligned}$$

Therefore

$$F(s) = \frac{s+2}{(s-2)^2} = \frac{1}{s-2} + \frac{4}{(s-2)^2} \tag{32.139}$$

Inverting the transform,

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{1}{s-2} + \frac{4}{(s-2)^2}\right] \tag{32.140}$$
$$= \mathcal{L}^{-1}\left[\frac{1}{s-2}\right] + 4\mathcal{L}^{-1}\left[\frac{1}{(s-2)^2}\right] \tag{32.141}$$
$$= e^{2t}\mathcal{L}^{-1}\left[\frac{1}{s}\right] + 4e^{2t}\mathcal{L}^{-1}\left[\frac{1}{t^2}\right] \tag{32.142}$$
$$= e^{2t} \cdot 1 + 4e^{2t} \cdot t \tag{32.143}$$
$$= (1+t)e^{2t}] \qquad \square \tag{32.144}$$

Example 32.14. Solve $y' + 4y = e^{-4t}$, $y(0) = 2$, using Laplace Transforms. Applying the transform,

$$\mathcal{L}[y'] + 4\mathcal{L}[y] = \mathcal{L}\left[e^{-4t}\right] \tag{32.145}$$
$$sY(s) - y(0) + 4Y(s) = \frac{1}{s+4} \tag{32.146}$$
$$(s+4)Y(s) = 2 + \frac{1}{s+4} \tag{32.147}$$

Solving for $Y(s)$,

$$Y(s) = \frac{2}{s+4} + \frac{1}{(s+4)^2} \tag{32.148}$$
$$y(t) = \mathcal{L}^{-1}[Y(s)] \tag{32.149}$$
$$= \mathcal{L}^{-1}\left[\frac{2}{s+4} + \frac{1}{(s+4)^2}\right] \tag{32.150}$$
$$= \mathcal{L}^{-1}\left[\frac{2}{s+4}\right] + \mathcal{L}^{-1}\left[\frac{1}{(s+4)^2}\right] \tag{32.151}$$
$$= 2e^{-4t}\mathcal{L}^{-1}\left[\frac{1}{s}\right] + e^{-4t}\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] \tag{32.152}$$
$$= 2e^{-4t} \cdot 1 + e^{-4t} \cdot t \tag{32.153}$$
$$= (2+t)e^{-4t} \qquad \square \tag{32.154}$$

Summary of Translation Formulas

$$\mathcal{L}[\mathcal{U}(t-a)] = \frac{e^{-as}}{s} \tag{32.155}$$

$$\mathcal{L}[f(t)\mathcal{U}(t-a)] = e^{-sa}\mathcal{L}[f(t+a)] \tag{32.156}$$

$$\mathcal{L}[f(t-a)\mathcal{U}(t-a)] = e^{-as}F(s) \tag{32.157}$$

$$\mathcal{L}\left[e^{at}f(t)\right] = F(s-a) \tag{32.158}$$

$$\mathcal{L}^{-1}[F(s-a)] = e^{at}f(t) \tag{32.159}$$

Convolution

Definition 32.12 (Convolution)**.** Let $f(t)$ and $g(t)$ be integrable functions on $(0, \infty)$. Then the **convolution of f and g** is defined by

$$(f * g)(t) = \int_0^t f(u)g(t-u)dt \tag{32.160}$$

Som Useful Properties of the Convolution

1. $f * g = g * f$ (commutative)
2. $f * (g + h) = f * g + f * h$ (distributive)
3. $f * (g * h) = (f * g) * h$ (associative)
4. $f * 0 = 0 * f = 0$ (convolution with zero is zero)

Example 32.15. Find $\sin t * \cos t$

$$\sin t * \cos t = \int_0^t \sin x \cos(t-x)dx \tag{32.161}$$

$$= \int_0^t \sin x(\cos t \cos x + \sin t \sin x)dx \tag{32.162}$$

$$= \cos t \int_0^t \sin x \cos x dx + \sin t \int_0^t \sin^2 x dx \tag{32.163}$$

$$= \cos t \, \frac{1}{2}\sin^2 x \Big|_0^t + \sin t \left(\frac{x}{2} - \frac{\sin 2x}{4}\right)\Big|_0^t \tag{32.164}$$

$$= \frac{1}{2}\cos t \sin^2 t + \sin t \left(\frac{t}{2} - \frac{\sin 2t}{4}\right) \tag{32.165}$$

Using trig substitution,

$$\sin t * \cos t = \frac{1}{2}\cos t \sin^2 t + \frac{1}{2}\sin t\,(t - \sin t \cos t) \tag{32.166}$$

$$= \frac{1}{2} t \sin t \qquad \square \tag{32.167}$$

Theorem 32.13 (Convolution Theorem)**.** Let $f(t)$ and $g(t)$ be piecewise continuous functions on $[0, \infty)$ of expnonential order with Laplace Transforms $F(s)$ and $G(s)$. Then

$$\mathcal{L}[f * g] = F(s)G(s) \tag{32.168}$$

The Laplace transform of the convolution is the product of the transforms. In its inverse form, the inverse transform of the product is the convolution:

$$\mathcal{L}^{-1}[F(s)G(s)] = f * g \tag{32.169}$$

Proof.

$$F(s)G(s) = \left(\int_0^\infty f(t)e^{-ts}dt\right)\left(\int_0^\infty g(x)e^{-sx}dx\right) \tag{32.170}$$

$$= \int_0^\infty \int_0^\infty f(t)g(x)e^{-ts-xs}dtdx \tag{32.171}$$

$$= \int_0^\infty f(t)\left(\int_0^\infty g(x)e^{-s(t+x)}dx\right)dt \tag{32.172}$$

In the inner integral let $u = t + x$. Then

$$F(s)G(s) = \int_0^\infty f(t)\left(\int_t^\infty g(u-t)e^{-su}du\right)dt \tag{32.173}$$

Interchanging the order of integration:

$$F(s)G(s) = \int_0^\infty e^{-su}\left(\int_0^t g(u-t)f(t)dt\right)du \tag{32.174}$$

$$= \int_0^\infty e^{-su}(f * g)(u)du \tag{32.175}$$

$$= \mathcal{L}[f * g] \tag{32.176}$$

as expected. $\square$

Example 32.16. Find a function $f(t)$ whose Laplace Transform is

$$F(s) = \frac{1}{s(s+1)} \tag{32.177}$$

The inverse transform is

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] && (32.178)\\ &= \mathcal{L}^{-1}\left[\frac{1}{s}\right]\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] && (32.179)\\ &= 1\cdot e^{-1\cdot t} && (32.180)\\ &= e^{-t} \quad \square && (32.181)\end{aligned}$$

Example 32.17. Find a function $f(t)$ whose Laplace Transform is

$$F(s) = \frac{s}{(s^2+4s-5)^2} \tag{32.182}$$

The inverse transform is

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left[\frac{s}{(s^2+4s-5)^2}\right] && (32.183)\\ &= \mathcal{L}^{-1}\left[\frac{s}{s^2+4s-5}\cdot\frac{1}{s^2+4s-5}\right] && (32.184)\\ &= \mathcal{L}^{-1}\left[\frac{s}{(s+5)(s-1)}\cdot\frac{1}{(s+5)(s-1)}\right] && (32.185)\\ &= \mathcal{L}^{-1}\left[\frac{s}{(s+5)(s-1)}\right]\cdot\mathcal{L}^{-1}\left[\frac{1}{(s+5)(s-1)}\right] && (32.186)\end{aligned}$$

Using partial fractions,

$$\frac{1}{(s+5)(s-1)} = -\frac{1}{6}\cdot\frac{1}{s+5}+\frac{1}{6}\cdot\frac{1}{s-1} \tag{32.187}$$

$$\frac{s}{(s+5)(s-1)} = \frac{5}{6}\cdot\frac{1}{s+5}+\frac{1}{6}\cdot\frac{1}{s-1} \tag{32.188}$$

The inverse transforms of (32.187) and (32.188) are thus

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{1}{(s+5)(s-1)}\right] &= -\frac{1}{6}\mathcal{L}^{-1}\left[\frac{1}{s+5}\right]+\frac{1}{6}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] && (32.189)\\ &= -\frac{1}{6}e^{-5t}+\frac{1}{6}e^{t} && (32.190)\end{aligned}$$

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{s}{(s+5)(s-1)}\right] &= \frac{5}{6}\mathcal{L}^{-1}\left[\frac{1}{s+5}\right]+\frac{1}{6}\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] && (32.191)\\ &= \frac{5}{6}e^{5t}+\frac{1}{6}e^{t} && (32.192)\end{aligned}$$

Substituting back into (32.186),

$$f(t) = \left(-\frac{1}{6}e^{-5t} + \frac{1}{6}e^{t}\right)\left(\frac{5}{6}e^{5t} + \frac{1}{6}e^{t}\right) \quad \square \tag{32.193}$$

Periodic Functions

Definition 32.14. A function $f(t)$ is said to be **periodic with period T** if there exists some positive number such that

$$f(t+T) = f(t) \tag{32.194}$$

for all values of t.

To find the transform of a periodic function we only have to integrate over a single interval, as illustrated below. Let $f(t)$ be a periodic function with period T. Then its transform is

$$F(S) = \int_0^\infty f(t)e^{-ts}dt \tag{32.195}$$

$$= \int_0^t f(t)e^{-st}dt + \int_T^\infty f(t)e^{-ts}dt \tag{32.196}$$

In the second integral make the substitution $u = t - T$, so that

$$F(S) = \int_0^t f(t)e^{-st}dt + \int_0^\infty f(u+T)e^{-(u+T)s}du \tag{32.197}$$

$$= \int_0^t f(t)e^{-st}dt + e^{-sT}\int_0^\infty f(u)e^{-us}du \qquad \text{since } f(u+T) = f(u) \tag{32.198}$$

$$= \int_0^t f(t)e^{-st}dt + e^{-sT}F(s) \tag{32.199}$$

Rearranging and solving for $F(s)$,

$$F(s)(1-e^{-sT}) = \int_0^t f(t)e^{-st}dt \tag{32.200}$$

which gives us the following result for periodic functions:

$$\boxed{F(s) = \mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}}\int_0^t f(t)e^{-st}dt} \tag{32.201}$$

Example 32.18. Find the Laplace transform of the full-wave rectification of $\sin t$,

$$f(t) = \begin{cases} \sin t & 0 \leq t < \pi \\ f(t-\pi) & t \geq \pi \end{cases} \tag{32.202}$$

Using the formula for the transform of a periodic function,

$$F(s) = \frac{1}{1-e^{-\pi s}} \int_0^\pi \sin t e^{-st} dt \tag{32.203}$$

$$= \frac{1}{1-e^{-\pi s}} \frac{1}{1+s^2} e^{-st}(-s\sin t - \cos t)\Big|_0^\pi \tag{32.204}$$

using (A.105) to find the integral. Hence

$$F(s) = \frac{1}{1-e^{-\pi s}} \frac{1}{1+s^2} [e^{-\pi s}(-s\sin \pi - \cos \pi) - e^0(-s\sin 0 - \cos 0)] \tag{32.205}$$

$$= \frac{1}{1-e^{-\pi s}} \frac{1}{1+s^2} [e^{-\pi s}(1) + (1)] \tag{32.206}$$

$$= \frac{1+e^{-\pi s}}{1-e^{-\pi s}} \cdot \frac{1}{1+s^2} \quad \square \tag{32.207}$$

Example 32.19. Solve $y'' + 9y = \cos 3t, y(0) = 2, y'(0) = 5$.

The Laplace Transform is

$$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = \frac{s}{s^2+9} \tag{32.208}$$

Substituting the initial conditions and grouping,

$$(9+s^2)Y(s) - 2s - 5 = \frac{s}{s^2+9} \tag{32.209}$$

rearranging terms and solving for $Y(s)$,

$$(9+s^2)Y(s) = 2s + 5 + \frac{s}{s^2+9} \tag{32.210}$$

$$Y(s) = \frac{2s+5}{9+s^2} + \frac{s}{(9+s^2)^2} \tag{32.211}$$

$$= 2 \cdot \frac{s}{9+s^2} + 5 \cdot \frac{1}{9+s^2} + \frac{s}{(9+s^2)^2} \tag{32.212}$$

The inverse transform is

$$y(t) = 2 \cdot \mathcal{L}^{-1}\left[\frac{s}{9+s^2}\right] + 5 \cdot \mathcal{L}^{-1}\left[\frac{1}{9+s^2}\right] + \mathcal{L}^{-1}\left[\frac{s}{(9+s^2)^2}\right] \tag{32.213}$$

$$= \frac{2}{3}\cos 3t + \frac{5}{3}\sin 3t + \mathcal{L}^{-1}\left[\frac{s}{9+s^2}\right] * \mathcal{L}^{-1}\left[\frac{1}{9+s^2}\right] \tag{32.214}$$

$$= \frac{2}{3}\cos 3t + \frac{5}{3}\sin 3t + \frac{1}{9}\cos 3t * \sin 3t \tag{32.215}$$

The convolution is

$$\cos 3t * \sin 3t = \int_0^t \cos 3x \sin 3(t-x) dx \tag{32.216}$$

$$= \frac{1}{2} t \sin(3t) \tag{32.217}$$

hence

$$y(t) = \frac{2}{3}\cos 3t + \frac{5}{3}\sin 3t + \frac{1}{18} t \sin 3t \quad \square \tag{32.218}$$

Impulses

Impulses are short, sudden perturbations of a system: quickly tapping on the accelerator of your car, flicking a light switch on and off, injection of medicine into the bloodstream, etc. It is convenient to describe these by box functions – with a step on followed by a step off. We define the **Unit Impulse of Width a at the Origin** by the function

$$\delta_a(t) = \begin{cases} 0 & t < -a \\ 2a & -a \le t < a \\ 0 & t > a \end{cases} \tag{32.219}$$

In terms of the unit step function,

$$\delta_a(t) = \frac{1}{2a}(\mathcal{U}(t+a) - \mathcal{U}(t-a)) \tag{32.220}$$

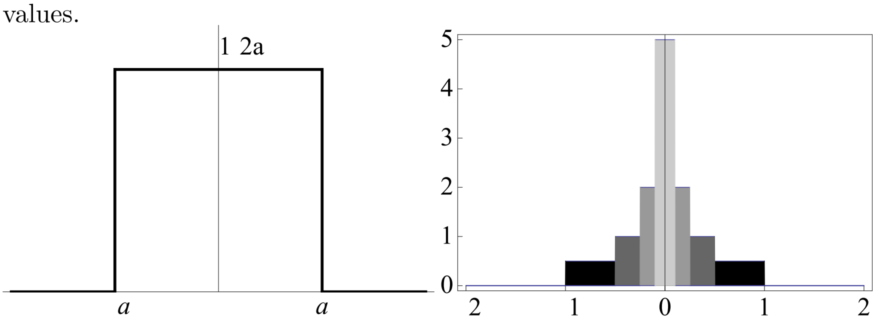
As the value of a is decreased the width of the box gets narrower but the height increases, making it much more of a sudden spike, but in each case, the area of the box is unity. In the limit, a sequence of narrower and narrower boxes approaches an infinitely tall spike which we call the **Dirac-delta function**[2]

$$\delta(t) = \lim_{a \to 0} \delta_a(t) \tag{32.221}$$

[2] For Paul Dirac (1902-1982), one of the founders of quantum mechanics.

Similarly, we can express a unit impulse centered at t_0 by a right shift, as $\delta_a(t-t_0)$, and an infinite unite spike as $\delta(t-t_0)$.

Figure 32.6: Unit impulse of width a at the origin (left). Sequence of sharper and sharper impulses on the right, with smaller and smaller a values.



The Laplace transform of the unit impulse is

$$
\begin{aligned}
\mathcal{L}[\delta_a(t-t_0)] &= \mathcal{L}\left[\frac{1}{2a}(\mathcal{U}(t+a-t_0)-\mathcal{U}(t-a-t_0))\right] && (32.222)\\
&= \frac{1}{2a}\left[\mathcal{L}[\mathcal{U}(t+a-t_0)]-\mathcal{L}[\mathcal{U}(t-a-t_0)]\right] && (32.223)\\
&= \frac{e^{(a-t_0)s}}{2as}-\frac{e^{-(a+t_0)s}}{2as} && (32.224)\\
&= e^{-t_0 s}\frac{e^{as}-e^{-as}}{2as} && (32.225)\\
&= e^{-t_0 s}\frac{\sinh as}{as} && (32.226)
\end{aligned}
$$

To get the Laplace transform of the delta function we take the limit as $a \to 0$,

$$
\mathcal{L}[\delta(t-t_0)] = \lim_{a\to 0} e^{-t_0 s}\frac{\sinh as}{as} \tag{32.227}
$$

Since the right hand side $\to 0/0$ we can use L'Hopital's rule from calculus,

$$
\begin{aligned}
\mathcal{L}[\delta(t-t_0)] &= e^{-t_0 s}\lim_{a\to 0}\frac{a\cosh as}{a} && (32.228)\\
&= e^{-t_0 s}\lim_{a\to 0}\cosh as && (32.229)\\
&= e^{-t_0 s} && (32.230)
\end{aligned}
$$

Since when $t_0 = 0$, $e^{-t_0 s} = e^0 = 1$,

$$\mathcal{L}[\delta(t)] = 1 \tag{32.231}$$

Example 32.20. Solve the initial value problem

$$y'' + y = \delta(y - \pi),\ y(0) = y'(0) = 0 \tag{32.232}$$

The Laplace transform gives us

$$s^2 Y(s) - sy(0) - y'(0) + Y(S) = e^{-\pi s} \tag{32.233}$$

$$(s^2 + 1)Y(s) = e^{-\pi s} \tag{32.234}$$

$$Y(s) = \frac{e^{-\pi s}}{1 + s^2} \qquad = \tag{32.235}$$

But recall that (see (32.156))

$$\mathcal{L}[f(t-a)\mathcal{U}(t-a)] = e^{-as}F(s) \tag{32.236}$$

$$\implies f(t-a)\mathcal{U}(t-a) = \mathcal{L}^{-1}\left[e^{-as}F(s)\right] \tag{32.237}$$

$$\implies f(t-\pi)\mathcal{U}(t-\pi) = \mathcal{L}^{-1}\left[e^{-a\pi}F(s)\right] \tag{32.238}$$

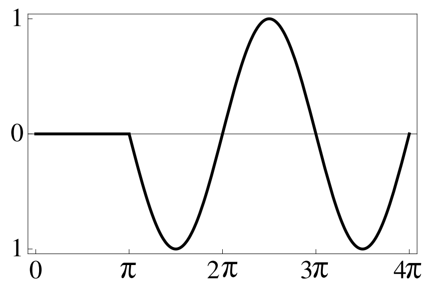
If we let $F(s) = 1/(1+s^2)$ then $f(t) = \sin t$. Hence

$$\mathcal{U}(t-\pi)\sin t = \mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{1+s^2}\right] \tag{32.239}$$

and therefore

$$y(t) = \mathcal{U}(t-\pi)\sin t \tag{32.240}$$

Figure 32.7: Solution of example in equation 32.232. The spring is initially quiescent because there is neither any initial displacement nor velocity, but at $t = \pi$ there is a unit impulse applied causing the spring to begin oscillations.



Lesson 33

Numerical Methods

Euler's Method

By a **dynamical system** we will mean a system of **differential equations** of the form

$$\left.\begin{aligned} y_1' &= f_1(t, y_1, y_2, \dots, y_n) \\ y_2' &= f_2(t, y_1, y_2, \dots, y_n) \\ &\vdots \\ y_n' &= f_n(t, y_1, y_2, \dots, y_n) \end{aligned}\right\} \tag{33.1}$$

and accompanying **initial conditions**

$$y_1(t_0) = y_{10}, y_2(t_0) = y_{2,0}, \dots, y_n(t_0) = y_{n0} \tag{33.2}$$

In the simplest case we have a single differential equation and initial condition (n=1)

$$y' = f(t, y), \;\; y(0) = y_0 \tag{33.3}$$

While it is possible to define dynamical systems that cannot be expressed in this form, e.g., they have partial derivatives or depend on function values at earlier time points, we will confine our study to these equations. In particular we will look as some of the techniques to solve the single equation 33.3. The generalization to higher dimensions (more equations) comes from replacing all of the variables in our methods with vectors.

Programming languages, in general, do not contain methods to solve differential equations, although there are large, freely available libraries that can be used for this purpose. Analysis environments like Mathematica and

Matlab have an extensive number of functions built in for this purpose, so in general, you will never actually need to implement the methods we will discuss in the next several sections if you confine yourself to those environments. Sometimes, however, the built-in routines don't provide enough generality and you will have to go in and modify them. In this case it helps to understand the basics of the numerical solution of differential equations.

By a *numerical solution* of the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0 \tag{33.4}$$

we mean a sequence of values

$$y_0, y_1, y_2, ..., y_{n-1}, y_n; \tag{33.5}$$

a corresponding *mesh* or *grid* $\mathcal{M}$ by

$$\mathcal{M} = \{t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n\}; \tag{33.6}$$

and a *grid spacing* as

$$h_j = t_{j+1} - t_j \tag{33.7}$$

Then the *numerical solution* or *numerical approximation to the solution* is the sequence of points

$$(t_0, y_0), (t_1, y_1), \ldots, (t_{n-1}, y_{n-1}), (t_n, y_n) \tag{33.8}$$

In this solution the point (t_j, y_j) represents the numerical approximation to the solution point $y(t_j)$. We can imagine plotting the points (33.8) and then "connecting the dots" to represent an approximate image of the graph of $y(t), t_0 \leq t \leq t_n$. We will use the convenient notation

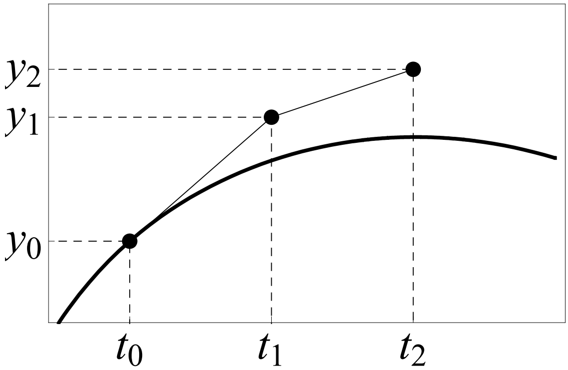
$$y_n \approx y(t_n) \tag{33.9}$$

which is read as "y_n is the numerical approximation to $y(t)$ at $t = t_n$."

Euler's Method or the **Forward Euler's Method** is constructed as illustrated in figure 33.1. At grid point t_n, $y(t) \approx y_n$, and the slope of the solution is given by exactly $y' = f(t_n, y(t_n))$. If we approximate the slope by the straight line segment between the numerical solution at t_n and the numerical solution at t_{n+1} then

$$y_n'(t_n) \approx \frac{y_{n+1} - y_n}{t_{n+1} - t_n} = \frac{y_{n+1} - y_n}{h_n} \tag{33.10}$$

Figure 33.1: Illustration of Euler's Method. A tangent line with slope $f(t_0, y_0)$ is constructed from (t_0, y_0) forward a distance $h = t_1 - t_0$ in the $t-$ direction to determined y_1. Then a line with slope $f(t_1, y_1)$ is constructed forward from (t_1, y_1) to determine y_2, and so forth. Only the first line is tangent to the actual solution; the subsequent lines are only approximately tangent.



Since $y'(t) = f(t, y)$, we can approximate the left hand side of (33.10) by

$$y_n'(t_n) \approx f(t_n, y_n) \tag{33.11}$$

and hence

$$\boxed{y_{n+1} = y_n + h_n f(t_n, y_n)} \tag{33.12}$$

It is often the case that we use a fixed *step size* $h = t_{j+1} - t_j$, in which case we have

$$t_j = t_0 + jh \tag{33.13}$$

In this case the Forward Euler's method becomes

$$y_{n+1} = y_n + hf(t_n, y_n) \tag{33.14}$$

The Forward Euler's method is sometimes just called *Euler's Method.*

An alternate derivation of equation (33.12) is to expand the solution $y(t)$ in a Taylor Series about the point $t = t_n$:

$$\begin{aligned} y(t_{n+1}) = y(t_n + h_n) &= y(t_n) + h_n y'(t_n) + \frac{h_n^2}{2} y''(t_n) + \cdots && (33.15)\\ &= y(t_n) + h_n f(t_n, y(_n)) + \cdots && (33.16) \end{aligned}$$

We then observe that since $y_n \approx y(t_n)$ and $y_{n+1} \approx y(t_{n+1})$, then (33.12) follows immediately from (33.16).

If the scalar initial value problem of equation (33.4) is replaced by a systems of equations

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}), \qquad \mathbf{y}(t_0) = \mathbf{y}_0 \tag{33.17}$$

then the Forward Euler's Method has the obvious generalization

$$\mathbf{y}_{n+1} = \mathbf{y}n + h\mathbf{f}(t_n, \mathbf{y}_n) \tag{33.18}$$

Example 33.1. Solve $y' = y, y(0) = 1$ on the interval [0, 1] using $h = 0.25$.

The exact solution is $y = e^x$. We compute the values using Euler's method. For any given time point t_k, the value y_k depends purely on the values of t_{k_1} and y_{k_1}. This is often a source of confusion for students: although the formula $y_{k+1} = y_k + hf(t_k, y_k)$ only depends on t_k and not on t_{k+1} it gives the value of y_{k+1}.

We are given the following information:

$$\left.\begin{aligned} (t_0, y_0) &= (0, 1) \\ f(t, y) &= y \\ h &= 0.25 \end{aligned}\right\} \tag{33.19}$$

We first compute the solution at $t = t_1$.

$$y_1 = y_0 + hf(t_0, y_0) = 1 + (0.25)(1) = 1.25 \tag{33.20}$$

$$t_1 = t_0 + h = 0 + 0.25 = 0.25 \tag{33.21}$$

$$(t_1, y_1) = (0.25, 1.25) \tag{33.22}$$

Then we compute the solutions at $t = t_1, t_2, \ldots$ until $t_{k+1} = 1$.

$$y_2 = y_1 + hf(t_1, y_1) \tag{33.23}$$

$$= 1.25 + (0.25)(1.25) = 1.5625 \tag{33.24}$$

$$t_2 = t_1 + h = 0.25 + 0.25 = 0.5 \tag{33.25}$$

$$(t_2, y_2) = (0.5, 1.5625) \tag{33.26}$$

$$y_3 = y_2 + hf(t_2, y_2) \tag{33.27}$$

$$= 1.5625 + (0.25)(1.5625) = 1.953125 \tag{33.28}$$

$$t_3 = t_2 + h = 0.5 + 0.25 = 0.75 \tag{33.29}$$

$$(t_3, y_3) = (0.75, 1.953125) \tag{33.30}$$

$$y_4 = y_3 + hf(t_3, y_3) \tag{33.31}$$
$$= 1.953125 + (0.25)(1.953125) = 2.44140625 \tag{33.32}$$
$$t_4 = t_3 + 0.25 = 1.0 \tag{33.33}$$
$$(t_4, y_4) = (1.0, 2.44140625) \tag{33.34}$$

Since $t_4 = 1$ we are done. The solutions are tabulated in table **??** for this and other step sizes. □

t	$h = 1/2$	$h = 1/4$	$h = 1/8$	$h = 1/16$	exact solution
0.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.0625				1.0625	1.0645
0.1250			1.1250	1.1289	1.1331
0.1875				1.1995	1.2062
0.2500		1.2500	1.2656	1.2744	1.2840
0.3125				1.3541	1.3668
0.3750			1.4238	1.4387	1.4550
0.4375				1.5286	1.5488
0.5000	1.5000	1.5625	1.6018	1.6242	1.6487
0.5625				1.7257	1.7551
0.6250			1.8020	1.8335	1.8682
0.6875				1.9481	1.9887
0.7500		1.9531	2.0273	2.0699	2.1170
0.8125				2.1993	2.2535
0.8750			2.2807	2.3367	2.3989
0.9375				2.4828	2.5536
1.0000	2.2500	2.4414	2.5658	2.6379	2.7183

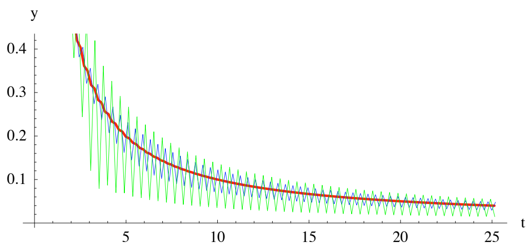
The Backwards Euler Method

Now consider the IVP

$$y' = -5ty^2 + \frac{5}{t} - \frac{1}{t^2}, \quad y(1) = 1 \tag{33.35}$$

The exact solution is $y = 1/t$. The numerical solution is plotted for three different step sizes on the interval $[1, 25]$ in the following figure. Clearly something appears to be happening here around $h = 0.2$, but what is it? For smaller step sizes, a relatively smooth solution is obtained, and for larger values of h the solution becomes progressively more jagged (figure **??**).

Figure 33.2: Solutions for equation 33.35 for various step sizes using Euler's method.

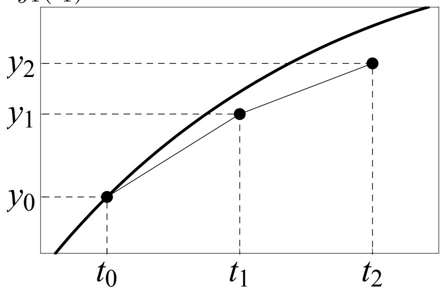


This example illustrates a problem that occurs in the solution of differential equations, known as **stiffness**. Stiffness occurs when the numerical method becomes unstable. One solution is to modify Euler's method as illustrated in figure 33.3 to give the **Backward's Euler Method**:

$$\boxed{y_n = y_{n-1} + h_n f(t_n, y_n)} \tag{33.36}$$

The problem with the Backward's Euler method is that we need to know the answer to compute the solution: y_n exists on both sides of the equation, and in general, we can not solve explicitly for it. The Backwards Euler Method is an example of an **implicit method**, because it contains y_n implicitly. In general it is not possible to solve for y_n explicitly as a function of y_{n-1} in equation 33.36, even though it is sometimes possible to do so for specific differential equations. Thus at each mesh point one needs to make some first guess to the value of y_n and then perform some additional refinement

Figure 33.3: Illustration of the Backward's Euler Method. Instead of constructing a tangent line with slope $f(t_0, y_0)$ through (t_0, y_0) a line with slope $f(t_1, y_1)$ is constructed. This necessitates knowing the solution at the t_1 in order to determine $y_1(t_1)$



to improve the calculation of y_n before moving on to the next mesh point. A common method is to use **fixed point iteration** on the equation

$$y = k + hf(t, y) \tag{33.37}$$

where $k = y_{n-1}$. The technique is summarized here:

- Make a first guess at y_n and use that in right hand side of 33.36. A common first guess that works reasonably well is

$$y_n^{(0)} = y_{n-1} \tag{33.38}$$

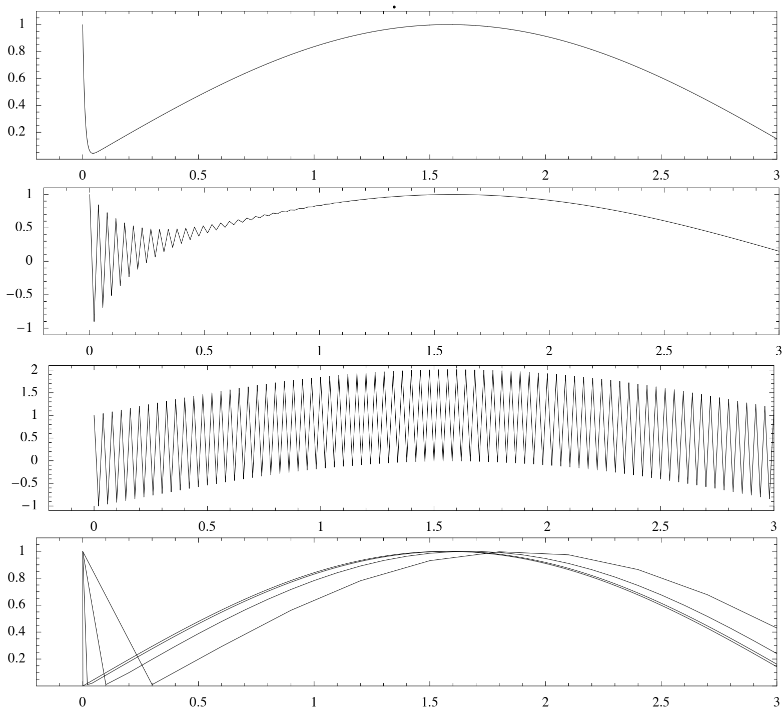
- Use the better estimate of y_n produced by 33.36 and then evaluate 33.36 again to get a third guess, e.g.,

$$y_n^{(\nu+1)} = y_{n-1} + hf(t_n, y_n^{(\nu)}) \tag{33.39}$$

- Repeat the process until the difference between two successive guesses is smaller than the desired tolerance.

It turns out that Fixed Point iteration will only converge if there is some number $K < 1$ such that $|\partial g/\partial y| < K$ where $g(t, y) = k + hf(t, y)$. A more stable method technique is Newton's method.

Figure 33.4: Result of the forward Euler method to solve $y' = -100(y - \sin t), y(0) = 1$ with $h = 0.001$ (top), $h = 0.019$ (middle), and $h = 0.02$ (third). The bottom figure shows the same equation solved with the backward Euler method for step sizes of $h = 0.001, 0.02, 0.1, 0.3$, left to right curves, respectively



Improving Euler's Method

All numerical methods for initial value problems of the form

$$y'(t) = f(t, y), \quad y(t_0) = y_0 \tag{33.40}$$

variations of the form

$$y_{n+1} = y_n + \phi(t_n, y_n, \dots) \tag{33.41}$$

for some function ϕ. In Euler's method, $\phi = hf(t_n, y_n)$; in the Backward's Euler method, $\phi = hf(t_{n+1}, y_{n+1})$. In general we can get a more accurate

result with a smaller step size. However, in order to reduce computation time, it is desirable to find methods that will give better results withou a significant decrease in step size. We can do this by making ϕ depend on values of the solution at multiple time points. For example, a **Linear Multistep Method** has the form

$$y_{n+1} + a_0 y_n + a_1 y_{n-1} + \cdots = h(b_0 f_{n+1} + b_1 f_n + b_2 f_{n-1} + \cdots) \quad (33.42)$$

For some numbers $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$. Euler's method has $a_0 = -1, a_1 = a_2 = \cdots = 0$ and $b_1 = 1, b_0 = b_2 = b_3 = \cdots = 0$

Here we introduce the **Local Truncation Error**, one measure of the "goodness" of a numerical method. The Local truncation error tells us the error in the calculation of y, in units of h, at each step t_n assuming that there we know y_{n-1} precisely correctly. Suppose we have a numerical estimate y_n of the correct solution at $y(t_n)$. Then the Local Truncation Error is defined as

$$\begin{aligned} \text{LTE} &= \frac{1}{h}(y(t_n) - y_n) && (33.43) \\ &= \frac{1}{h}(y(t_n) - y(t_{n-1}) + y(t_{n-1}) - y_n) && (33.44) \end{aligned}$$

Assuming we know the answer precisely correctly at t_{n-1} then we have

$$y_{n-1} = y(t_{n-1}) \quad (33.45)$$

so that

$$\begin{aligned} \text{LTE} &= \frac{y(t_n) - y(t_{n-1})}{h} + \frac{y_{n-1} - y_n}{h} && (33.46) \\ &= \frac{y_(t_n) - y(t_{n-1})}{h} - \frac{1}{h}\phi(t_n, y_n, \ldots) && (33.47) \end{aligned}$$

For Euler's method,

$$\phi = hf(t, y) \quad (33.48)$$

hence

$$\text{LTE(Euler)} = \frac{y(t_n) - y(t_{n-1})}{h} - f(t_n, y_n) \quad (33.49)$$

If we expand y in a Taylor series about t_{n-1},

$$y(t_n) = y(t_{n-1}) + hy'(t_{n-1}) + \frac{h^2}{2}y''(t_{n-1}) + \cdots \tag{33.50}$$

$$= y(t_{n-1}) + hf(t_{n-1}, y_{n-1}) + \frac{h^2}{2}y''(t_{n-1}) + \cdots \tag{33.51}$$

Thus

$$\text{LTE(Euler)} = \frac{h}{2}y''(t_{n-1}) + c_2h^2 + c_3h^3 + \cdots \tag{33.52}$$

for some constants $c_1, c_2, ...$ Because the lowest order term in powers of h is proportional to h, we say that

$$\text{LTE(Euler)} = O(h) \tag{33.53}$$

and say that Euler's method is a **First Order Method**. In general, to improve accuracy for a given step size, we look for higher order methods, which are $O(h^n)$; the larger the value of n, the better the method in general.

The **Trapezoidal Method** averages the values of f at the two end points. It has an iteration formula given by

$$y_n = y_{n-1} + \frac{h_n}{2}\left(f(t_n, y_n) + f(t_{n-1}, y_{n-1})\right) \tag{33.54}$$

We can find the LTE as follows by expanding the Taylor Series,

$$\text{LTE(Trapezoidal)} = \frac{y(t_n) - y(t_{n-1})}{h} - f(t_n, y_n) \tag{33.55}$$

$$= \frac{1}{h}\left(y(t_{n-1}) + hy'(t_{n-1}) + \frac{h^2}{2}y''(t_{n-1}) + \frac{h^3}{3!}y'''(t_{n-1}) + \cdots - y(t_{n-1})\right)$$
$$- \frac{1}{2}\left(f(t_n, y_n) + f(t_{n-1}, y_{n-1})\right) \tag{33.56}$$

Therefore using $y'(t_{n-1}) = f(t_{n-1}, y_{n-1})$,

$$\text{LTE(Trapezoidal)} = \frac{1}{2}f(t_{n-1}, y_{n-1}) + \frac{h}{2}y''(t_{n-1}) + \frac{h^2}{6}y'''(t_{n-1}) \cdots - \frac{1}{2}f(t_n, y_n) \tag{33.57}$$

Expanding the final term in a Taylor series,

$$\begin{aligned}
f(t_n, y_n) &= y'(t_n) && (33.58)\\
&= y'(t_{n-1}) + hy''(t_{n-1}) + \frac{h^2}{2}y'''(t_{n-1}) + \cdots && (33.59)\\
&= f(t_{n-1}, y_{n-1}) + hy''(t_{n-1}) + \frac{h^2}{2}y'''(t_{n-1}) + \cdots && (33.60)
\end{aligned}$$

Therefore the Trapezoidal method is a second order method:

$$\begin{aligned}
\text{LTE(Trapezoidal} &= \frac{1}{2}f_{n-1} + \frac{h}{2}y''_{n-1} + \frac{h^2}{6}y'''_{n-1} + \cdots \\
&\qquad - \frac{1}{2}f_{n-1} - \frac{1}{2}hy''_{n-1} - \frac{1}{4}h^2y'''_{n-1} + \cdots && (33.61)\\
&= -\frac{1}{12}h^2y'''_{n-1} + \cdots && (33.62)\\
&= O(h^2) && (33.63)
\end{aligned}$$

The **theta method** is given by

$$y_n = y_{n-1} + h\left[\theta f(t_{n-1}, y_{n-1}) + (1-\theta)f(t_n, y_n)\right] \tag{33.64}$$

The theta method is implicit except when $\theta = 1$, where it reduces to Euler's method, and is first order unless $\theta = 1/2$. For $\theta = 1/2$ it becomes the trapezoidal method. The usefulness of the comes from the ability to remove the error for specific high order terms. For example, when $\theta = 2/3$, there is no h^3 term even though there is still an h^2 term. This can help if the coefficient of the h^3 is so larger that it overwhelms the the h^2 term for some values of h.

The second-order **midpoint method** is given by

$$y_n = y_{n-1} + h_n f\left(t_{n-1/2}, \frac{1}{2}[y_n + y_{n-1}]\right) \tag{33.65}$$

The **modified Euler Method**, which is also second order, is

$$y_n = y_{n-1} + \frac{h_n}{2}\left[f(t_{n-1}, y_{n-1}) + f(t_n, y_{n-1} + hf(t_{n-1}, y_{n-1}))\right] \tag{33.66}$$

Heun's Method is

$$y_n = y_{n-1}+\frac{h_n}{4}\left[f(t_{n-1},y_{n-1})+3f\left(t_{n-1}+\frac{2}{3}h, y_{n-1}+\frac{2}{3}hf(t_{n-1},y_{n-1})\right)\right] \tag{33.67}$$

Both Heun's method and the modified Euler method are second order and are examples of two-step Runge-Kutta methods. It is clearer to implement these in two "stages," eg., for the modified Euler method,

$$\tilde{y}_n = y_{n-1} + hf(t_{n-1}, y_{n-1}) \tag{33.68}$$

$$y_n = y_{n-1} + \frac{h_n}{2}[f(t_{n-1}, y_{n-1}) + f(t_n, \tilde{y}_n)] \tag{33.69}$$

while for Heun's method,

$$\tilde{y}_n = y_{n-1} + \frac{2}{3}hf(t_{n-1}, y_{n-1}) \tag{33.70}$$

$$y_n = y_{n-1} + \frac{h_n}{4}\left[f(t_{n-1}, y_{n-1}) + 3f\left(t_{n-1} + \frac{2}{3}h, \tilde{y}_n\right)\right] \tag{33.71}$$

Runge-Kutta Fourth Order Method. This is the "gold standard" of numerical methods - its a lot higher order than Euler but still really easy to implement. Other higher order methods tend to be very tedious – even to code – although once coded they can be very useful. Four intermediate calculations are performed at each step:

$$k_1 = hf(t_n, y_n) \tag{33.72}$$

$$k_2 = hf(t_n + .5h, y_n + .5k_1) \tag{33.73}$$

$$k_3 = hf(t_n + .5h, y_n + .5k_2) \tag{33.74}$$

$$k_4 = hf(t_n + h, y_n + k_3) \tag{33.75}$$

Then the subsequent iteration is given by

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{33.76}$$

Example 33.2. Compute the solution to the test equation $y' = y, y(0) = 1$ on $[0, 1]$ using the 4-stage Runge Kutta method with $h = 1/2$.

Since we start at $t = 0$ and need to compute through $t = 1$ we have to compute two iterations of RK. For the first iteration,

$$k_1 = y_0 = 1 \tag{33.77}$$

$$k_2 = y_0 + \frac{h}{2} f(t_0, k_1) \tag{33.78}$$
$$= 1 + (0.25)(1) \tag{33.79}$$
$$= 1.25 \tag{33.80}$$

$$k_3 = y_0 + \frac{h}{2} f(t_{1/2}, k_2) \tag{33.81}$$
$$= 1 + (0.25)(1.25) \tag{33.82}$$
$$= 1.3125 \tag{33.83}$$

$$k_4 = y_0 + h f(t_{1/2}, k_3) \tag{33.84}$$
$$= 1 + (0.5)(1.3125) \tag{33.85}$$
$$= 1.65625 \tag{33.86}$$

$$y_1 = y_0 + \frac{h}{6}(f(t_0, k_1) + 2f(t_{1/2}, k_2) + 2f(t_{1/2}, k_3) + f(t_1, k_4)) \tag{33.87}$$
$$= y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{33.88}$$
$$= 1 + \frac{.5}{6} 1 + 2(1.25) + 2(1.3125) + 1.65625 \tag{33.89}$$
$$= 1.64844 \tag{33.90}$$

Thus the numerical approximation to $y(0.5)$ is $y_1 \approx 1.64844$. For the second step,

$$k_1 = y_1 = 1.64844 \tag{33.91}$$

$$k_2 = y_1 + \frac{h}{2} f(t_1, k_1) \tag{33.92}$$
$$= 1.64844 + (0.25)(1.64844) \tag{33.93}$$
$$= 2.06055 \tag{33.94}$$

$$k_3 = y_1 + \frac{h}{2} f(t_{1.5}, k_2) \tag{33.95}$$
$$= 1.64844 + (0.25)(2.06055) \tag{33.96}$$
$$= 2.16358 \tag{33.97}$$

$$k_4 = y_1 + hf(t_{1.5}, k_3) \tag{33.98}$$
$$= 1.64844 + (0.5)(2.16358) \tag{33.99}$$
$$= 2.73023 \tag{33.100}$$

$$y_2 = y_1 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{33.101}$$
$$= 1.64844 + \frac{.5}{6}1.64844 + 2(2.06055) + 2(2.16358) + 2.73023 \tag{33.102}$$
$$= 2.71735 \tag{33.103}$$

This gives us a numerical approximation of $y(1) \approx 2.71735$, and error of approximately 0.034% (the exact value is $e \approx 2.71828$. By comparison, a forward Euler computation with the same step size will yield a numerical result of 2.25, an error approximately 17%. $\square$

Since it is an explicit method, the Runge-Kutta 4-stage method is very easy to implement in a computer, even though calculations are very tedious to do by hand

Lesson 34

Critical Points of Autonomous Linear Systems

Definition 34.1. A differential equation (or system of differential equations) is called **autonomous** if it does not expressly depend on the independent variable t, e.g., the equation $y' = f(t, y)$ can be replaced with $y' = g(y)$ for some function g.

Example 34.1. The function $y' = \sin y$ is autonomous, while the function $y' = t \cos y$ is not autonomous. The system

$$x' = \cos y + x^2 \tag{34.1}$$
$$y' = \sin x \tag{34.2}$$

is autonomous, while the system

$$x' = \cos y + x^2 \tag{34.3}$$
$$y' = \sin x + e^{at} \tag{34.4}$$

is not autonomous. □

When we talk about systems, we do not loose any generality by only focusing on autonomous systems because any non-autonomous system can be converted to an autonomous system with one additional variable. For example, the system 34.3 to 34.4 can be made autonomous by defining a new

variable with differential equation $z = t$, with differential equation $z' = 1$, and adding the third equation to the system:

$$x' = \cos y + x^2 \tag{34.5}$$
$$y' = \sin x + e^{az} \tag{34.6}$$
$$z' = 1 \tag{34.7}$$

We will focus on the general two dimensional autonomous system

$$\begin{aligned} x' &= f(x,y) \quad x(t_0) = x_0 \\ y' &= g(x,y) \quad y(t_0) = y_0 \end{aligned} \tag{34.8}$$

where f and g are differentiable functions of x and y in some open set containing the point (x_0, y_0). From the uniqueness theorem, we know that there is precisely one solution $\{x(t), y(t)\}$ to (34.8). We can plot this solution as a curve that goes through the point (x_0, y_0) and extends in either direction for some distance. We call this curve the **trajectory** of the solution. The xy-plane itself we will call the **phase-plane**.

We must take care to distinguish trajectories from solutions: *the trajectory is a curve* in the xy plane, while *the solution is a set of points that are marked by time.* Different solutions follow the same trajectory, but at different times. The difference can be illustrated by the following analogy. Imagine that you drive from home to school every day on certain road, say Nordhoff Street. Every day you drive down Nordhoff from the 405 freeway to Reseda Boulevard. On Mondays and Wednesdays, you have morning classes and have to drive this path at 8:20 AM. On Tuesdays and Thursdays you have evening classes and you drive the same path, moving in the same direction, but at 3:30 PM. Then Nordhoff Street is your trajectory. You follow two different solutions: one which puts you at the 405 at 8:20 AM and another solution that puts at the 405 at 3:30 PM. Both solutions follow the same trajectory, but at different times.

Returning to differential equations, an autonomous system with initial conditions $x(t_0) = a, y(t_0) = b$ will have the same trajectory as an autonomous system with initial conditions $x(t_1) = a, y(t_1) = b$, but it will be a different solution because it will be at different points along the trajectory at different times.

In any set where $f(x,y) \neq 0$ we can form the differential equation

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} \tag{34.9}$$

Since both f and g are differentiable, their quotient is differentiable (away from regions where$f = 0$) and hence Lipshitz; consequently the initial value problem

$$\frac{dy}{dx} = \frac{g(x,y)}{f(x,y)}, \quad y(x_0) = y_0 \tag{34.10}$$

has a unique solution, which corresponds to the trajectory of equation (34.8) through the point (x_0, y_0). Since the solution is unique, we conclude that there is only one trajectory through any point, except possibly where$f(x,y) = 0$. A plot showing the one-parameter family of solutions to (34.9), annotated with arrows to indicate the direction of motion in time, is called a **phase portrait** of the system.

Example 34.2. Let $\{x_1, y_1\}$ and $\{x_2, y_2\}$ be the solutions of

$$x' = -y, \;\; x(0) = 1 \tag{34.11}$$

$$y' = x, \;\; y(0) = 0 \tag{34.12}$$

and

$$x' = -y, \;\; x(\pi/4) = 1 \tag{34.13}$$

$$y' = x, \;\; y(\pi/4) = 0 \tag{34.14}$$

respectively.

The solutions are different, but both solutions follow the same trajectory.

To see this we solve the system by forming the second order differential equation representing this system: differentiate the second equation to obtain $y'' = x'$; then substitute the first equation to obtain

$$y'' = -y \tag{34.15}$$

The characteristic equation is $r^2+1 = 0$ with roots of $\pm i$; hence the solutions are linear combinations of sines and cosines. As we have seen, the general solution to this system is therefore

$$y = A\cos t + B\sin t \tag{34.16}$$

$$x = -A\sin t + B\cos t \tag{34.17}$$

where the second equation is obtained by differentiating the first (because $x = y'$).

The initial conditions for (34.11) give

$$0 = A\cos 0 + B\sin 0 = A \tag{34.18}$$

$$1 = -A\sin 0 + B\cos 0 = B \tag{34.19}$$

hence the solution is

$$y = \sin t \tag{34.20}$$

$$x = \cos t \tag{34.21}$$

The trajectory is the unit circle because

$$x^2 + y^2 = 1 \tag{34.22}$$

for all t, and the solution is the set of all points starting at an angle of 0 from the x axis.

The initial conditions for (34.13), on the other hand, give

$$0 = A\frac{\sqrt{2}}{2} + B\frac{\sqrt{2}}{2} \tag{34.23}$$

$$1 = -A\frac{\sqrt{2}}{2} + B\frac{\sqrt{2}}{2} \tag{34.24}$$

adding the two equations

$$1 = 2B\frac{\sqrt{2}}{2} = B\sqrt{2} \implies B = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \tag{34.25}$$

hence

$$A = -B = -\frac{\sqrt{2}}{2} \tag{34.26}$$

which gives a solution of

$$y = -\frac{\sqrt{2}}{2}\cos t + \frac{\sqrt{2}}{2}\sin t \tag{34.27}$$

$$x = \frac{\sqrt{2}}{2}\sin t + \frac{\sqrt{2}}{2}\cos t \tag{34.28}$$

Using the $\cos\pi/4 = \sqrt{2}/2$ and angle addition formulas,

$$y = -\cos\frac{\pi}{4}\cos t + \sin\frac{\pi}{4}\sin t \tag{34.29}$$
$$= -\cos\left(\frac{\pi}{4} + t\right) \tag{34.30}$$
$$x = \cos\frac{\pi}{4}\sin t + \sin\frac{\pi}{4}\cos t \tag{34.31}$$
$$= \sin\left(\frac{\pi}{4} + t\right) \tag{34.32}$$

The trajectory is also the the unit circle, because we still have $x^2 + y^2 = 1$, but not the solution starts at the point 45 degrees above the x-axis.

The two solutions are different, but they both follow the same trajectory. □

Definition 34.2. A **Critical Point** of the system $x' = f(x, y), y' = g(x, y)$ (or **fixed point** or **local equilibrium**) is a any point (x^*, y^*) such that both of the following conditions

$$f(x^*, y^*) = 0 \tag{34.33}$$
$$g(x^*, y^*)0 \tag{34.34}$$

hold simultaneously.

A critical point is a unique kind of trajectory: anything that starts there, stays there, for all time. They are thus zero-dimensional, isolated trajectories. Furthermore, no other solution can pass through a critical point, because once there, it would have to stop.[1]

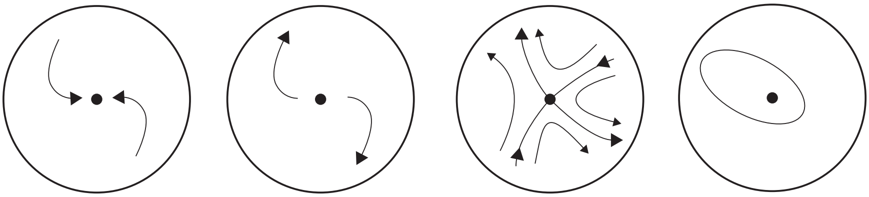
If there is an open neighborhood about a critical point that does not contain any other critical points, it is called an **isolated critical point**. Certain properties (that we will discuss presently) of linear systems at isolated critical points determine the global geometry of the phase portrait; for nonlinear systems, these same properties determine the local geometry.

We will classify a critical point **P** based on the dynamics of a particle placed in some small neighborhood N of **P**, and observe what happens as t increases. We call P a

1. **Sink**, **Attractor**, or **Stable Node** if all such points move toward P;

[1] This does not prevent critical points from being limit points of solutions (as$t \to \pm\infty$) and thus appearing to be part of another trajectory but this is an artifact of how we draw phase portraits.

Figure 34.1: Varieties of isolated critical points of linear systems. The circles represented neighborhoods; the lines trajectories; the arrows the direction of change with increasing value of t in a solution.



2. **Source**, **Repellor**, or **Unstable Node** if all such points move away from P;

3. **Center**, if all trajectories loop around P in closed curves;

4. **Saddle Point** or **Saddle Node** if some solutions move toward P and some move away.

Let us begin by studying the linear system

$$\left.\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}\right\} \tag{34.35}$$

where the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{34.36}$$

is nonsingular (i.e., its determinant is non-zero and the matrix is invertible). To find the critical points we set the derivatives equal to zero:

$$0 = ax^* + by^* \tag{34.37}$$

$$0 = cx^* + dy^* \tag{34.38}$$

The only solution is $(x^*, y^*) = (0, 0)$. Hence (34.35) has a single isolated critical point at the origin.

The solution depends on the roots of the characteristic equation, or eigen-

values, of the matrix. We find these from the determinant of $A - \lambda I$,

$$0 = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} \tag{34.39}$$

$$= (a-\lambda)(d-\lambda) - bc \tag{34.40}$$

$$= \lambda^2 - (a+d)\lambda + ad - bc \tag{34.41}$$

Writing

$$T = \text{trace}(A) = a + d \tag{34.42}$$

$$\Delta = \det(A) = ad - bc \tag{34.43}$$

the characteristic equation becomes

$$0 = \lambda^2 - T\lambda + \Delta \tag{34.44}$$

There are two roots,

$$\lambda_1 = \frac{1}{2}\left(T + \sqrt{T^2 - 4\Delta}\right) \tag{34.45}$$

$$\lambda_2 = \frac{1}{2}\left(T - \sqrt{T^2 - 4\Delta}\right) \tag{34.46}$$

$$\tag{34.47}$$

If there are two linearly independent eigenvectors $\mathbf{v}$ and $\mathbf{w}$, then the general solution of the linear system is

$$\begin{pmatrix} x \\ y \end{pmatrix} = A\mathbf{v}e^{\lambda_1 t} + B\mathbf{w}e^{\lambda_2 t} \tag{34.48}$$

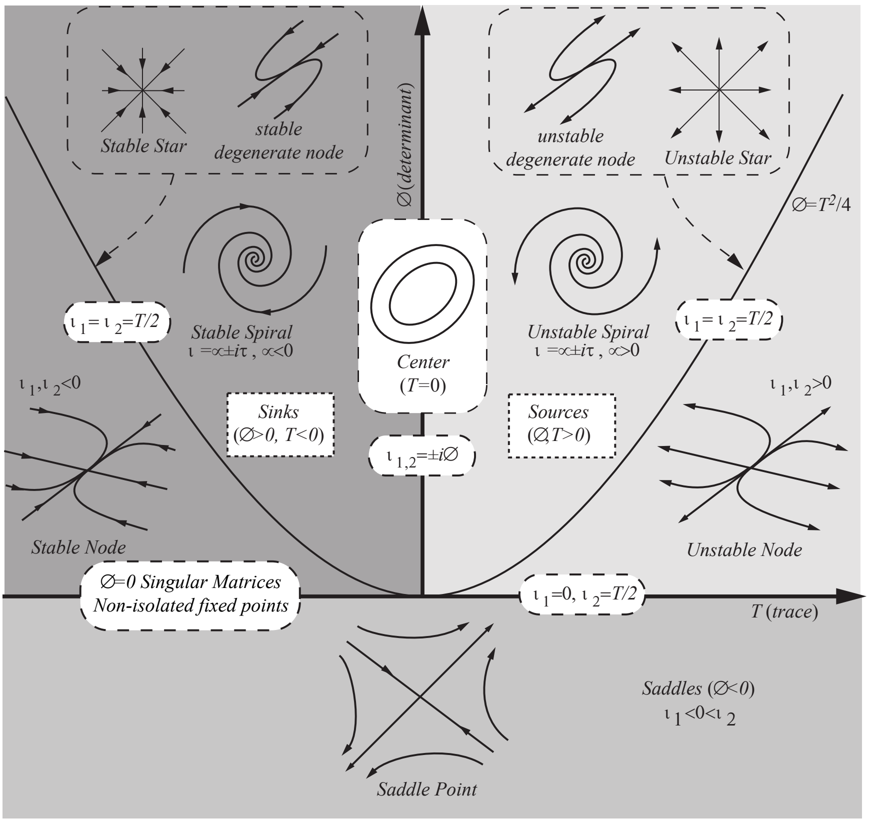
This result holds even if the eigenvalues are a complex conjugate pair, or if there is a degenerate eigenvalue with multiplicity 2, so long as there are a pair of linearly independent eigenvectors.

If the eigenvalue is repeated but has only one eigenvector, $\mathbf{v}$ then

$$\begin{pmatrix} x \\ y \end{pmatrix} = \left[A\mathbf{v} + B(t\mathbf{v} + \mathbf{w})\right] e^{\lambda t} \tag{34.49}$$

where $\mathbf{w}$ is the generalized eigenvector. In the following paragraphs we will study the implications of equations (34.48) to (34.49) for various values of the eigenvalues, as determined by the values of the trace and determinant.

Figure 34.2: Topology of critical points as determined by the trace and determinant of a linear (or linearized) system.



Distinct Real Nonzero Eigenvalues of the Same Sign

If $T^2 > 4\Delta > 0$ both eigenvalues will be real and distinct. Repeated eigenvalues are excluded because $T^2 \neq 4\Delta$.

If$T > 0$, both eigenvalues will both be positive, while if $T < 0$ both eigenvalues will be negative (note that $T = 0$ does not fall into this category). The solution is given by (34.48); A and B are determined by initial conditions.

The special case $A = B = 0$ occurs only when$x(t_0) = y(t_0) = 0$, which gives the isolated critical point at the origin. For nonzero initial conditions, the solution will be a linear combination of the two **eigensolutions**

$$y_1 = \mathbf{v}e^{\lambda_1 t} \tag{34.50}$$

$$y_2 = \mathbf{w}e^{\lambda_2 t} \tag{34.51}$$

By convention we will choose λ_1 to be the larger of the two eigenvalues in magnitude; then we will call the directions parallel to $\mathbf{v}$ and $\mathbf{v}$the **fast eigendirection** and the **slow eigendirection**, respectively.

If both eigenvalues are positive, every solution becomes unbounded as $t \to \infty$ (because $e^{\lambda_i t} \to \infty$ as $t \to \infty$) and approaches the origin as $t \to -\infty$ (because $e^{\lambda_i t} \to 0$ as $t \to -\infty$), and the origin is called a **source**, **repellor**, or **unstable node**.

If both eigenvalues are negative, the situation is reversed: every solution approaches the origin in positive time, as $t \to \infty$, because $e^{\lambda_i t} \to 0$ as $t \to \infty$, and diverges in negative time as $t \to -\infty$ (because $e^{\lambda_i t} \to \infty$ at $t \to -\infty$), and the origin is called a **sink**, **attractor**, or **stable node**.

The names **stable node** and **unstable node** arise from the dynamical systems interpretation: a particle that is displaced an arbitrarily small distance away from the origin will move back towards the origin if it is a stable node, and will move further away from the origin if it is an unstable node.

Despite the fact that the trajectories **approach** the origin either as $t \to \infty$ or $t \to -\infty$, the only trajectory that **actually passes through the origin** is the isolated (single point) trajectory at the origin. Thus the only trajectory that passes through the origin is the one with $A = B = 0$. To see this consider the following. For a solution to intersect the origin at a time t would require

$$A\mathbf{v}e^{\lambda_1 t} + B\mathbf{w}e^{\lambda_2 t} = 0 \tag{34.52}$$

This means that we could define new constants at any fixed time t

$$a(t) = Ae^{\lambda_1 t} \text{ and } b(t) = Be^{\lambda_2 t} \tag{34.53}$$

such that

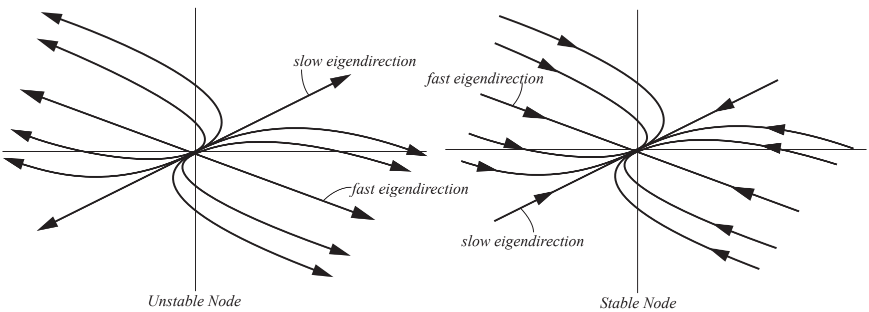
$$a(t)\mathbf{v} + b(t)\mathbf{w} = 0 \tag{34.54}$$

Since $mathbfv$ and $\mathbf{w}$ are linearly independent, the only way this can happen is when $a(t) = 0$ and $b(t) = 0$ at the same same time. There is no t value for which this can happen because

$$e^{\lambda t} \neq 0 \tag{34.55}$$

for all possible values of λ. Thus the only way for (34.54) to be true is for $A = B = 0$. This corresponds to the solution $(x, y) = (0, 0)$, which is the point at the origin.

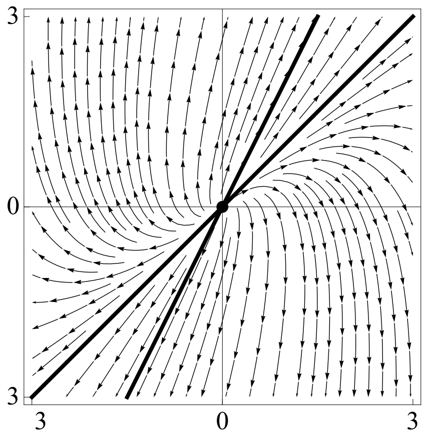
Figure 34.3: Phase portraits typical of an unstable (left) and stable (right) node.



The geometry is illustrated in figure 34.3. The two straight lines passing through the origin correspond to $A = 0, B \neq 0$ and $B = 0, A \neq 0$ respectively, namely the two eigendirections. The solutions on the eigendirections are $Ae^{\lambda_1 t}$ and $Be_2^{\lambda}t$, with different initial conditions represented by different values of A and B. If the eigenvalues are positive, a particle that starts along one of these eigendirections (i.e., has initial conditions that start the system on an eigendirection) moves in a straight line away from the origin as $t \to \infty$ and towards the origin as $t \to -\infty$. If the eigenvalues are negative, the particle moves in a straight line away from the origin as $t \to -\infty$ and towards the origin as$t \to \infty$. Trajectories that pass through other points have both $A \neq 0$ and $B \neq 0$ so that $\mathbf{y} = A\mathbf{v}e^{\lambda_1 t} + B\mathbf{w}e^{\lambda_2 t}$. If both eigenvalues are positive, then for large positive time (as $t \to \infty$) the fast

eigendirection $\{\lambda_1, \ \ v_1\}$dominates and the solutions will approach lines parallel to v_1, while for large negative time ($t \to -\infty$) the solutions approach the origin parallel to the slow eigendirection. The situation is reversed for negative eigenvalues: the trajectories approach the origin along the slow eigendirection as $t \to \infty$ and diverge parallel to the fast eigendirection as $t \to -\infty$.

Figure 34.4: Phase portrait for the system (34.56).



Example 34.3. Classify the fixed points and sketch the phase portrait of the system

$$\left.\begin{aligned} x' &= x + y \\ y' &= -2x + 4y. \end{aligned}\right\} \tag{34.56}$$

The matrix of coefficients is

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix} \tag{34.57}$$

so that $T = 5$ and $\Delta = 6$. Consequently the eigenvalues are

$$\lambda = \frac{1}{2}\left(T \pm \sqrt{T^2 - 4\Delta}\right) \tag{34.58}$$

$$= \frac{1}{2}\left(5 \pm \sqrt{25 - 24}\right) \tag{34.59}$$

$$= 3, 2 \tag{34.60}$$

The eigenvalues are real, positive, and distinct, so the fixed point is a source. The fast eigenvector, corresponding to the larger eigenvalue, $\lambda = 3$, is $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The slow eigenvector, corresponding to the smaller eigenvalue $\lambda = 2$, is $\mathbf{w} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Near the origin, the slow eigendirection dominates, while further away, the fast eigendirection dominates. The trajectories mostly leave the origin tangentially to the line $y = x$, which is along the slow eigendirection, then bend around parallel the fast eigendirection as one moves away from the origin. The phase portrait is illustrated in figure 34.4. $\square$

Repeated Real Nonzero Eigenvalues

If $T^2 = 4\Delta \neq 0$, the eigenvalues are real and repeated (we exclude the case with both eigenvalues equal to zero for now because that only occurs when the matrix of coefficients has a determinant of zero). In this case we are not required to have two linearly independent eigenvectors.

If there are two linearly independent eigenvectors $\mathbf{v}$ and $\mathbf{w}$, then the solution is

$$y = (A\mathbf{v} + Bt\mathbf{w})e^{\lambda t} \tag{34.61}$$

and if there is only a single eigenvector $\mathbf{v}$ then

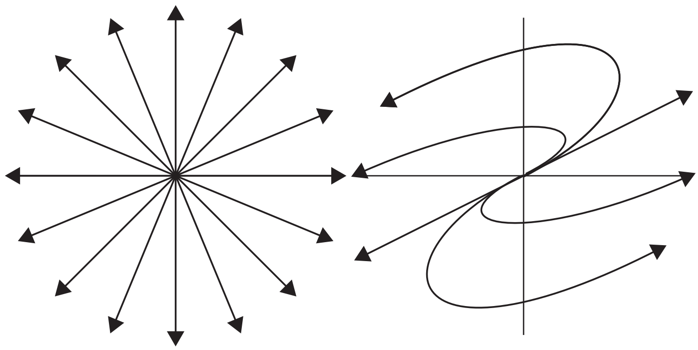
$$y = [Av + B(t\mathbf{v} + \mathbf{w})]e^{\lambda t} \tag{34.62}$$

where $\mathbf{w}$ is the generalized eigenvector satisfying $(A - \lambda I)\mathbf{w} = \mathbf{v}$.

In the first case (linearly independent eigenvectors) all solutions lie on straight lines passing through the origin, approaching the origin in positive time ($t \to \infty$) if $\lambda > 0$, and in negative time ($t \to -\infty$) if $\lambda < 0$. The critical point at the origin is called a **star node**: a **stable star** if $\lambda = T/2 < 0$ and an **unstable star** if $\lambda = T/2 > 0$.

If there is only a single eigenvector then the trajectories approach the origin tangent to $\mathbf{v}$; as one moves away from the origin, the trajectories bend around and diverge parallel to $\mathbf{v}$ (see figure 34.5). The origin is called either a **stable degenerate node** ($\lambda = T/2 < 0$) or an **unstable degenerate node** ($\lambda = T/2 > 0$).

Figure 34.5: Phase portraits typical of an unstable star node (left) and an unstable degenerate node (right). The corresponding stable nodes have the arrows pointing so that the solutions approach the origin in positive time.



Real Eigenvalues with Opposite Signs

If $\Delta < 0$, one eigenvalue will be positive and one eigenvalue will be negative, regardless of the value of T. Denote them as λ and $-\mu$, where $\lambda > 0$ and $\mu > 0$. The solution is

$$\mathbf{y} = A\mathbf{v}e^{\lambda t} + B\mathbf{w}e^{-\mu t} \tag{34.63}$$

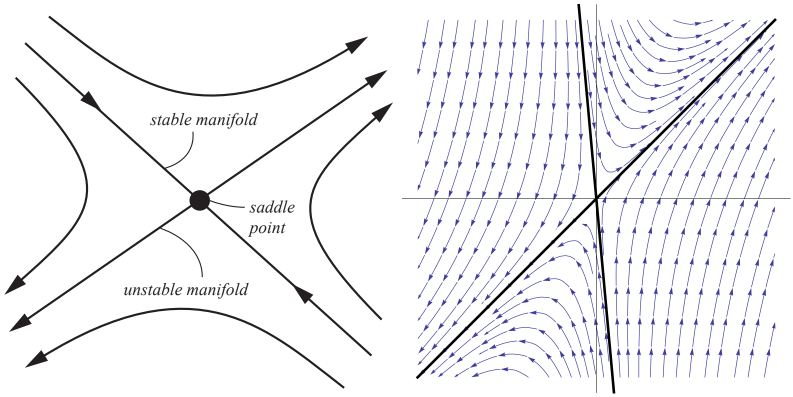
Solutions that start on the line through the origin with direction $\mathbf{v}$ ($A \neq 0$ but $B = 0$) diverge as $t \to \infty$ and approach the origin as $t \to -\infty$; the corresponding trajectory is called the **stable manifold of the critical point**.

Solutions that start on the line through the origin with direction $\mathbf{w}$ (($A = 0$ with $B \neq 0$) diverge as $t \to -\infty$ and approach the origin as $t \to \infty$; the corresponding trajectory is called the **unstable manifold of the critical point**. Besides the stable manifold and the unstable manifold, no other trajectories approach the origin. The critical point itself is called a **saddle point** or **saddle node** (see figure 34.6).

Example 34.4. The system

$$\left.\begin{aligned} x' &= 4x + y \\ y' &= 11x - 6y \end{aligned}\right\} \tag{34.64}$$

Figure 34.6: Topology of a saddle point (left) and phase portrait for example 34.4 (right).



has trace

$$T = 4 - 6 = -2 \tag{34.65}$$

and determinant

$$\Delta = (4)(-6) - (1)(11) = -24 + -11 = -35 \tag{34.66}$$

Thus

$$\sqrt{T^2 - 4\Delta} = \sqrt{(-2)^2 - 4(-35)} = \sqrt{144} = 12 \tag{34.67}$$

and the eigenvalues are

$$\lambda = \frac{1}{2}\left(T \pm \sqrt{T^2 - 4\Delta}\right) = \frac{-2 \pm 12}{2} = 5, -7 \tag{34.68}$$

Since the eigenvalues have different signs, the origin is a saddle point. Eigenvectors are $\begin{pmatrix}1\\1\end{pmatrix}$ (for 5) and $\begin{pmatrix}1\\-1\end{pmatrix}$ (for -7). The stable manifold is the line $y = x$ (corresponding to the negative eigenvalue), and the unstable manifold is the line $y = -11x$ (corresponding the positive eigenvalue). A phase portrait is shown in figure 34.6. $\square$

Complex Conjugate Pair with nonzero real part.

If $0 < T^2 < 4\Delta$ the eigenvalues will be a complex conjugate pair, with real part T. T may be either positive or negative, but $T = 0$ is excluded from this category (if $T = 0$, the eigenvalues will either be purely imaginary, when$\Delta > 0$, or real with opposite signs, if$\Delta < 0$).

Writing

$$\mu = T/2, \omega^2 = 4\Delta - T^2 \tag{34.69}$$

the eigenvalues become $\lambda = \mu \pm i\omega,\ \ \mu, \omega \in \mathbb{R}$. Designating the corresponding eigenvectors as $\mathbf{v}$ and $\mathbf{w}$, the solution is

$$\begin{aligned} y &= e^{\mu t}\left[A\mathbf{v}e^{i\omega t} + B\mathbf{w}e^{-i\omega t}\right] && (34.70)\\ &= e^{\mu t}\left[A\mathbf{v}\left(\cos\omega t + i\sin\omega t\right) + B\mathbf{w}\left(\cos\omega t - i\sin\omega t\right)\right] && (34.71)\\ &= e^{\mu t}\left[\mathbf{p}\cos\omega t + \mathbf{q}\sin\omega t\right] && (34.72)\end{aligned}$$

where

$$\begin{aligned} \mathbf{p} &= A\mathbf{v} + B\mathbf{w} && (34.73)\\ \mathbf{q} &= i(A\mathbf{v} - B\mathbf{w}) && (34.74)\end{aligned}$$

are purely real vectors (see exercise 6) for real initial conditions. The factor in parenthesis in (34.70) gives closed periodic trajectories in the xy plane, with period $2\pi/\omega$; the exponential factor modulates this parameterization with either a continually increasing ($\mu > 0$) or continually decreasing ($\mu < 0$) factor.

When $\mu > 0$, the solutions spiral away from the origin as $t \to \infty$ and in towards the origin as $t \to -\infty$. The origin is called an **unstable spiral**.

When $\mu < 0$, the solutions spiral away from the origin as $t \to -\infty$ and in towards the origin as $t \to \infty$; the origin is then called a **stable spiral**.

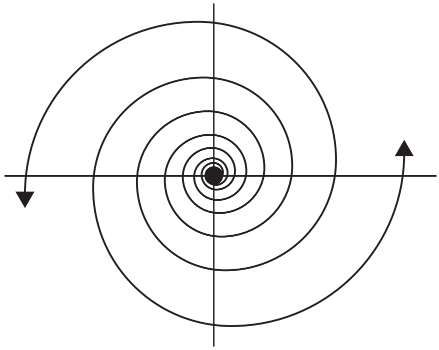
Example 34.5. The system

$$\left.\begin{aligned} x' &= -x + 2y\\ y' &= -2x - 3y \end{aligned}\right\} \tag{34.75}$$

has trace $T = -4$ and determinant $\Delta = 7$; hence the eigenvalues are

$$\lambda = \frac{1}{2}\left[T \pm \sqrt{T^2 - 4\Delta}\right] = -2 \pm i\sqrt{3} \tag{34.76}$$

Figure 34.7: An unstable spiral node.



which form a complex conjugate pair with negative real part. Hence the origin is a stable spiral center. □

Example 34.6. The system

$$\left.\begin{aligned} x' &= ax - y \\ y' &= x + ay \end{aligned}\right\} \tag{34.77}$$

where a is a small real number, has a spiral center at the origin. It is easily verified that the eigenvalues are $\lambda = a \pm i$. To get an explicit formula for the spiral we use the following identities:

$$rr' = xx' + yy' \tag{34.78}$$

$$r^2\theta' = xy' - yx' \tag{34.79}$$

to convert the system into polar coordinates. The radial variation is

$$\begin{aligned} rr' &= xx' + yy' && (34.80) \\ &= x(ax - y) + y(x + ay) && (34.81) \\ &= a(x^2 + y^2) && (34.82) \\ &= ar^2 && (34.83) \end{aligned}$$

and therefore (canceling a common factor of r from both sides of the equation),

$$r' = ar \tag{34.84}$$

The angular change is described by

$$
\begin{aligned}
r^2\theta' &= xy' - yx' && (34.85)\\
&= x(x+ay) - y(ax-y) && (34.86)\\
&= x^2 + y^2 && (34.87)\\
&= r^2 && (34.88)
\end{aligned}
$$

so that (canceling the common r^2 on both sides of the equation),

$$\theta' = 1 \tag{34.89}$$

Dividing r' by θ' gives

$$\frac{dr}{d\theta} = \frac{r'}{\theta'} = ar \tag{34.90}$$

and hence

$$r = r_0 e^{a(\theta-\theta_0)} \tag{34.91}$$

which is a logarithmic spiral with $r(t_0) = r_0$ and $\theta(t_0) = \theta_0$. $\square$

Purely Imaginary Eigenvalues

If $T = 0$ and $\Delta > 0$ the eigenvalues will be a purely imaginary conjugate pair $\lambda = \pm i\Delta$. The solution is

$$\mathbf{y} = \mathbf{v}\cos\omega t + \mathbf{w}\sin\omega t \tag{34.92}$$

The origin is called a ***center***. Center's have the unusual property (unusual compared to the other types of critical points we have discussed thus far) of being topologically unstable to variations in the equations, as illustrated by the following example. A system is topologically unstable if any small change in the system changes the geometry, e.g., the systems changes from one type of center to another.

Example 34.7. The system

$$\left.\begin{aligned} x' &= x + 2y \\ y' &= -3x - y \end{aligned}\right\} \tag{34.93}$$

has a trace of $T = 0$ and determinant of $\Delta = 5$. Thus

$$\lambda = \frac{1}{2}\left[T \pm \sqrt{T^2 - 4\Delta}\right] = \pm i\sqrt{5} \tag{34.94}$$

Since the eigenvalues are purely imaginary, the origin is a center.

If we perturb either of the diagonal coefficients, the eigenvalues develop a real part. For example, the system

$$\left.\begin{aligned} x' &= 1.01x + 2y \\ y' &= -3x - y \end{aligned}\right\} \tag{34.95}$$

has eigenvalues $\lambda \approx 0.005 \pm 2.23383i$, making it an unstable spiral; and the system

$$\left.\begin{aligned} x' &= 0.99x + 2y \\ y' &= -3x - y \end{aligned}\right\} \tag{34.96}$$

has eigenvalues $\lambda \approx -0.005 \pm 2.2383i$, for a stable spiral.

The magnitude of the real part grows approximately linearly as the perturbation grows. In general, the perturbed system

$$\left.\begin{aligned} x' &= (1+\varepsilon)x + 2y \\ y' &= -3x - y \end{aligned}\right\} \tag{34.97}$$

will have eigenvalues

$$\lambda = \frac{1}{2}\left[\varepsilon \pm \sqrt{-20 + 4\varepsilon + \varepsilon^2}\right] \tag{34.98}$$

$$= \frac{\varepsilon}{2} \pm i\sqrt{5}\sqrt{1 - \frac{\varepsilon}{5} - \frac{\varepsilon^2}{20}} \tag{34.99}$$

The results for perturbations of $\varepsilon = \pm 0.05$ are shown in figure 34.8. $\square$

Non-isolated Critical Points

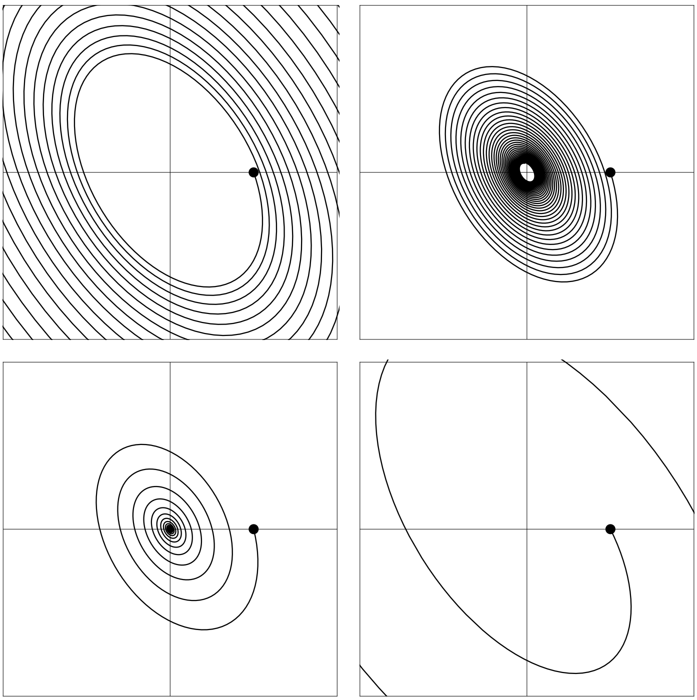
When the matrix $\mathbf{A}$ in equation (34.36) is singular, the critical points will not, in general, be isolated, and they will not fall into any of the categories that we have discussed so far.

Set $\Delta = ad - bc = 0$, and suppose that all four of a, b, c, and d are nonzero. Then we can solve for any one of the four coefficients in terms of the others, e.g., $d = bc/a$, so that the trajectories are defined by

$$\frac{dy}{dx} = \frac{cx + dy}{ax + by} = \frac{cx + (bc/a)y}{ax + by} = \frac{c}{a} = \frac{d}{b} \tag{34.100}$$

The trajectories are parallel lines with slope c/a. There is only one critical

Figure 34.8: Topological instability of center nodes. Solutions to equations (34.97) are plotted for *epsilon*=0.05, -0.05, -.25, 0.5, with initial conditions of $x, y = 1, 0$ (black dot). The bounding box is $[-2, 2] \times [-2, 2]$ in each case.



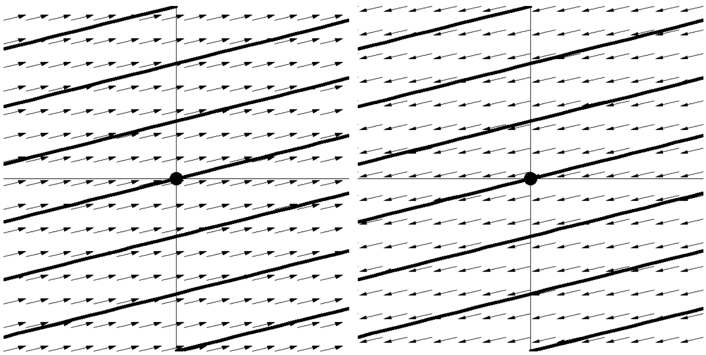
point, at the origin as usual. The nullclines are $y = -bx/a$ (for x) and $y = -(c/d)x$ for y (figure 34.9). At the other extreme, if all the coefficients are zero, then every point in the plane is a critical point and there are no trajectories – wherever you start, you will stay there for all time. If precisely one of a,b,c or d is zero, but the others are nonzero, the matrix will be nonsingular so the only remaining cases to consider have one or two coefficients nonzero.

If $a = b = 0$ and $c \neq 0$ and/or $d \neq 0$, we the system becomes

$$\left.\begin{aligned} x' &= 0 \\ y' &= cx + dy \end{aligned}\right\} \tag{34.101}$$

so the solutions are all vertical lines, and every point on the line $y = -cx/d$

Figure 34.9: Phase portraits of singular linear system where all coefficients are nonzero for $c/a > 0$.



is a critical point (when $d \neq 0$), or on the line $x = 0$ (when $d = 0$). The directions of motion along the trajectories switch along the critical line. A good analogy is to think of the critical line as the top of a ridge (or the bottom of a valley), compared to a single apex for a nonsingular system (Figure 34.9). We essentially have a whole line of sources or sinks.

By a similar argument, if $c = d = 0$ and $a \neq 0$ and/or $b \neq 0$, the system becomes

$$\left.\begin{aligned} x' &= ax + by \\ y' &= 0 \end{aligned}\right\} \tag{34.102}$$

The solutions are all horizontal lines, and there is a ridge or valley of critical points along the line $y = ax/b$ (if $b \neq 0$) or along the line $x = 0$ (if $b = 0$).

If $a = c = 0$ with $b \neq 0$ and $d \neq 0$, the system is

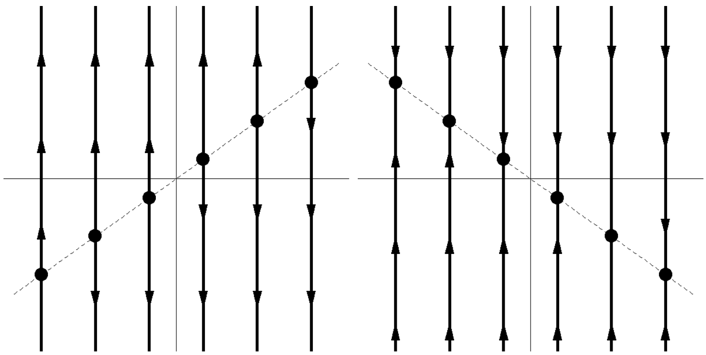
$$\left.\begin{aligned} x' &= by \\ y' &= dy \end{aligned}\right\} \tag{34.103}$$

The x-axis (the line $y = 0$) is a critical ridge (valley) and the trajectories are parallel lines with slope d/b. Similarly, if $b = d = 0$ with $a \neq 0$ and $d \neq 0$ the system is

$$\left.\begin{aligned} x' &= ax \\ y' &= ay \end{aligned}\right\} \tag{34.104}$$

so that the trajectories are parallel lines with slope c/a and the y-axis is a

Figure 34.10: phase portraits for the system (34.101); every point on the entire "ridge" line is a critical point.



critical ridge (valley).

Appendix A

Table of Integrals

Basic Forms

$$\int x^n dx = \frac{1}{n+1} x^{n+1} \tag{A.1}$$

$$\int \frac{1}{x} dx = \ln x \tag{A.2}$$

$$\int u dv = uv - \int v du \tag{A.3}$$

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \ln |ax+b| \tag{A.4}$$

Integrals of Rational Functions

$$\int \frac{1}{(x+a)^2} dx = -\frac{1}{x+a} \tag{A.5}$$

$$\int (x+a)^n dx = \frac{(x+a)^{n+1}}{n+1} + c, n \neq -1 \tag{A.6}$$

$$\int x(x+a)^n dx = \frac{(x+a)^{n+1}((n+1)x-a)}{(n+1)(n+2)} \tag{A.7}$$

$$\int \frac{1}{1+x^2}dx = \tan^{-1} x \tag{A.8}$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} \tag{A.9}$$

$$\int \frac{x}{a^2+x^2}dx = \frac{1}{2}\ln|a^2+x^2| + c \tag{A.10}$$

$$\int \frac{x^2}{a^2+x^2}dx = x - a\tan^{-1}\frac{x}{a} \tag{A.11}$$

$$\int \frac{x^3}{a^2+x^2}dx = \frac{1}{2}x^2 - \frac{1}{2}a^2\ln|a^2+x^2| \tag{A.12}$$

$$\int \frac{1}{ax^2+bx+c}dx = \frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} \tag{A.13}$$

$$\int \frac{1}{(x+a)(x+b)}dx = \frac{1}{b-a}\ln\frac{a+x}{b+x},\ a \neq b \tag{A.14}$$

$$\int \frac{x}{(x+a)^2}dx = \frac{a}{a+x} + \ln|a+x| \tag{A.15}$$

$$\begin{aligned}\int \frac{x}{ax^2+bx+c}dx &= \frac{1}{2a}\ln|ax^2+bx+c| \\ &\quad - \frac{b}{a\sqrt{4ac-b^2}}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}}\end{aligned} \tag{A.16}$$

Integrals with Roots

$$\int \sqrt{x-a}dx = \frac{2}{3}(x-a)^{3/2} \tag{A.17}$$

$$\int \frac{1}{\sqrt{x\pm a}}dx = 2\sqrt{x\pm a} \tag{A.18}$$

$$\int \frac{1}{\sqrt{a-x}}dx = -2\sqrt{a-x} \tag{A.19}$$

$$\int x\sqrt{x-a}dx = \frac{2}{3}a(x-a)^{3/2} + \frac{2}{5}(x-a)^{5/2} \tag{A.20}$$

$$\int \sqrt{ax+b}dx = \left(\frac{2b}{3a} + \frac{2x}{3}\right)\sqrt{ax+b} \tag{A.21}$$

$$\int (ax+b)^{3/2}dx = \frac{2}{5a}(ax+b)^{5/2} \tag{A.22}$$

$$\int \frac{x}{\sqrt{x\pm a}}dx = \frac{2}{3}(x\mp 2a)\sqrt{x\pm a} \tag{A.23}$$

$$\int \sqrt{\frac{x}{a-x}}dx = -\sqrt{x(a-x)} - a\tan^{-1}\frac{\sqrt{x(a-x)}}{x-a} \tag{A.24}$$

$$\int \sqrt{\frac{x}{a+x}}dx = \sqrt{x(a+x)} - a\ln\left[\sqrt{x} + \sqrt{x+a}\right] \tag{A.25}$$

$$\int x\sqrt{ax+b}dx = \frac{2}{15a^2}(-2b^2 + abx + 3a^2x^2)\sqrt{ax+b} \tag{A.26}$$

$$\begin{aligned}\int \sqrt{x(ax+b)}dx = \frac{1}{4a^{3/2}}&\left[(2ax+b)\sqrt{ax(ax+b)}\right.\\ &\left.-b^2\ln\left|a\sqrt{x} + \sqrt{a(ax+b)}\right|\right]\end{aligned} \tag{A.27}$$

$$\begin{aligned}\int \sqrt{x^3(ax+b)}dx = &\left[\frac{b}{12a} - \frac{b^2}{8a^2x} + \frac{x}{3}\right]\sqrt{x^3(ax+b)}\\ &+\frac{b^3}{8a^{5/2}}\ln\left|a\sqrt{x} + \sqrt{a(ax+b)}\right|\end{aligned} \tag{A.28}$$

$$\int \sqrt{x^2\pm a^2}dx = \frac{1}{2}x\sqrt{x^2\pm a^2} \pm \frac{1}{2}a^2\ln\left|x + \sqrt{x^2\pm a^2}\right| \tag{A.29}$$

$$\int \sqrt{a^2 - x^2} dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{1}{2} a^2 \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}} \tag{A.30}$$

$$\int x \sqrt{x^2 \pm a^2} dx = \frac{1}{3} \left(x^2 \pm a^2\right)^{3/2} \tag{A.31}$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln \left| x + \sqrt{x^2 \pm a^2} \right| \tag{A.32}$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} \tag{A.33}$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}} dx = \sqrt{x^2 \pm a^2} \tag{A.34}$$

$$\int \frac{x}{\sqrt{a^2 - x^2}} dx = -\sqrt{a^2 - x^2} \tag{A.35}$$

$$\int \frac{x^2}{\sqrt{x^2 \pm a^2}} dx = \frac{1}{2} x \sqrt{x^2 \pm a^2} \mp \frac{1}{2} a^2 \ln \left| x + \sqrt{x^2 \pm a^2} \right| \tag{A.36}$$

$$\begin{aligned} \int \sqrt{ax^2 + bx + c} dx &= \frac{b + 2ax}{4a} \sqrt{ax^2 + bx + c} \\ &+ \frac{4ac - b^2}{8a^{3/2}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx^+ c)} \right| \end{aligned} \tag{A.37}$$

$$\begin{aligned} \int x \sqrt{ax^2 + bx + c} = \frac{1}{48a^{5/2}} \Big(& 2\sqrt{a}\sqrt{ax^2 + bx + c} \\ & - \left(3b^2 + 2abx + 8a(c + ax^2)\right) \\ & + 3(b^3 - 4abc) \ln \left| b + 2ax + 2\sqrt{a}\sqrt{ax^2 + bx + x} \right| \Big) \end{aligned} \tag{A.38}$$

$$\int \frac{1}{\sqrt{ax^2 + bx + c}} dx = \frac{1}{\sqrt{a}} \ln \left| 2ax + b + 2\sqrt{a(ax^2 + bx + c)} \right| \tag{A.39}$$

$$\int \frac{x}{\sqrt{ax^2+bx+c}}dx = \frac{1}{a}\sqrt{ax^2+bx+c} - \frac{b}{2a^{3/2}} \ln\left|2ax+b+2\sqrt{a(ax^2+bx+c)}\right| \tag{A.40}$$

$$\int \frac{dx}{(a^2+x^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}} \tag{A.41}$$

Integrals with Logarithms

$$\int \ln ax dx = x \ln ax - x \tag{A.42}$$

$$\int \frac{\ln ax}{x} dx = \frac{1}{2}(\ln ax)^2 \tag{A.43}$$

$$\int \ln(ax+b)dx = \left(x+\frac{b}{a}\right)\ln(ax+b) - x, a \neq 0 \tag{A.44}$$

$$\int \ln\left(x^2+a^2\right)dx = x\ln\left(x^2+a^2\right) - 2x + 2a\tan^{-1}\frac{x}{a} \tag{A.45}$$

$$\int \ln\left(x^2-b^2\right)dx = x\ln\left(x^2-b^2\right) - 2x + a\ln\frac{x+a}{x-a} \tag{A.46}$$

$$\int \ln\left(ax^2+bx+c\right)dx = \frac{1}{a}\sqrt{4ac-b^2}\tan^{-1}\frac{2ax+b}{\sqrt{4ac-b^2}} - 2x + \left(\frac{b}{2a}+x\right)\ln\left(ax^2+bx+c\right) \tag{A.47}$$

$$\int x\ln(ax+b)dx = \frac{bx}{2a} - \frac{1}{4}x^2 + \frac{1}{2}\left(x^2-\frac{b^2}{a^2}\right)\ln(ax+b) \tag{A.48}$$

$$\int x \ln\left(a^2 - b^2 x^2\right) dx = -\frac{1}{2}x^2 + \frac{1}{2}\left(x^2 - \frac{a^2}{b^2}\right)\ln\left(a^2 - b^2x^2\right) \tag{A.49}$$

Integrals with Exponentials

$$\int e^{ax} dx = \frac{1}{a}e^{ax} \tag{A.50}$$

$$\begin{aligned}\int \sqrt{x} e^{ax} dx &= \frac{1}{a}\sqrt{x}e^{ax} + \frac{i\sqrt{\pi}}{2a^{3/2}}\operatorname{erf}\left(i\sqrt{ax}\right), \\ \text{where } \operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt\end{aligned} \tag{A.51}$$

$$\int x e^x dx = (x-1)e^x \tag{A.52}$$

$$\int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax} \tag{A.53}$$

$$\int x^2 e^x dx = \left(x^2 - 2x + 2\right)e^x \tag{A.54}$$

$$\int x^2 e^{ax} dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)e^{ax} \tag{A.55}$$

$$\int x^3 e^x dx = \left(x^3 - 3x^2 + 6x - 6\right)e^x \tag{A.56}$$

$$\int x^n e^{ax}\,\mathrm{d}x = \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,\mathrm{d}x \tag{A.57}$$

$$\begin{aligned}\int x^n e^{ax}\,\mathrm{d}x &= \frac{(-1)^n}{a^{n+1}}\Gamma[1+n, -ax], \\ \text{where } \Gamma(a,x) &= \int_x^\infty t^{a-1}e^{-t}\,\mathrm{d}t\end{aligned} \tag{A.58}$$

$$\int e^{ax^2}\,\mathrm{dx} = -\frac{i\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}\left(ix\sqrt{a}\right) \tag{A.59}$$

$$\int e^{-ax^2}\,\mathrm{dx} = \frac{\sqrt{\pi}}{2\sqrt{a}}\mathrm{erf}\left(x\sqrt{a}\right) \tag{A.60}$$

$$\int xe^{-ax^2}\,\mathrm{dx} = -\frac{1}{2a}e^{-ax^2} \tag{A.61}$$

$$\int x^2e^{-ax^2}\,\mathrm{dx} = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}\mathrm{erf}(x\sqrt{a}) - \frac{x}{2a}e^{-ax^2} \tag{A.62}$$

Integrals with Trigonometric Functions

$$\int \sin ax dx = -\frac{1}{a}\cos ax \tag{A.63}$$

$$\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a} \tag{A.64}$$

$$\int \sin^n ax dx = -\frac{1}{a}\cos ax \;\; {}_2F_1\left[\frac{1}{2}, \frac{1-n}{2}, \frac{3}{2}, \cos^2 ax\right] \tag{A.65}$$

$$\int \sin^3 ax dx = -\frac{3\cos ax}{4a} + \frac{\cos 3ax}{12a} \tag{A.66}$$

$$\int \cos ax dx = \frac{1}{a}\sin ax \tag{A.67}$$

$$\int \cos^2 ax dx = \frac{x}{2} + \frac{\sin 2ax}{4a} \tag{A.68}$$

$$\int \cos^p ax dx = -\frac{1}{a(1+p)}\cos^{1+p} ax \times {}_2F_1\left[\frac{1+p}{2}, \frac{1}{2}, \frac{3+p}{2}, \cos^2 ax\right] \tag{A.69}$$

$$\int \cos^3 ax dx = \frac{3\sin ax}{4a} + \frac{\sin 3ax}{12a} \tag{A.70}$$

$$\int \cos ax \sin bx dx = \frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, a \neq b \tag{A.71}$$

$$\int \sin^2 ax \cos bx dx = -\frac{\sin[(2a-b)x]}{4(2a-b)} + \frac{\sin bx}{2b} - \frac{\sin[(2a+b)x]}{4(2a+b)} \tag{A.72}$$

$$\int \sin^2 x \cos x dx = \frac{1}{3} \sin^3 x \tag{A.73}$$

$$\begin{aligned}\int \cos^2 ax \sin bx dx = & \frac{\cos[(2a-b)x]}{4(2a-b)} - \frac{\cos bx}{2b} \\ & - \frac{\cos[(2a+b)x]}{4(2a+b)}\end{aligned} \tag{A.74}$$

$$\int \cos^2 ax \sin ax dx = -\frac{1}{3a} \cos^3 ax \tag{A.75}$$

$$\begin{aligned}\int \sin^2 ax \cos^2 bx dx = & \frac{x}{4} - \frac{\sin 2ax}{8a} - \frac{\sin[2(a-b)x]}{16(a-b)} \\ & + \frac{\sin 2bx}{8b} - \frac{\sin[2(a+b)x]}{16(a+b)}\end{aligned} \tag{A.76}$$

$$\int \sin^2 ax \cos^2 ax dx = \frac{x}{8} - \frac{\sin 4ax}{32a} \tag{A.77}$$

$$\int \tan ax dx = -\frac{1}{a} \ln \cos ax \tag{A.78}$$

$$\int \tan^2 ax dx = -x + \frac{1}{a} \tan ax \tag{A.79}$$

$$\int \tan^n ax dx = \frac{\tan^{n+1} ax}{a(1+n)} \times {}_2F_1\left(\frac{n+1}{2}, 1, \frac{n+3}{2}, -\tan^2 ax\right) \tag{A.80}$$

$$\int \tan^3 axdx = \frac{1}{a}\ln\cos ax + \frac{1}{2a}\sec^2 ax \tag{A.81}$$

$$\int \sec xdx = \ln|\sec x + \tan x| = 2\tanh^{-1}\left(\tan\frac{x}{2}\right) \tag{A.82}$$

$$\int \sec^2 axdx = \frac{1}{a}\tan ax \tag{A.83}$$

$$\int \sec^3 x\,\mathrm{dx} = \frac{1}{2}\sec x\tan x + \frac{1}{2}\ln|\sec x + \tan x| \tag{A.84}$$

$$\int \sec x\tan xdx = \sec x \tag{A.85}$$

$$\int \sec^2 x\tan xdx = \frac{1}{2}\sec^2 x \tag{A.86}$$

$$\int \sec^n x\tan xdx = \frac{1}{n}\sec^n x, n \neq 0 \tag{A.87}$$

$$\int \csc xdx = \ln\left|\tan\frac{x}{2}\right| = \ln|\csc x - \cot x| \tag{A.88}$$

$$\int \csc^2 axdx = -\frac{1}{a}\cot ax \tag{A.89}$$

$$\int \csc^3 xdx = -\frac{1}{2}\cot x\csc x + \frac{1}{2}\ln|\csc x - \cot x| \tag{A.90}$$

$$\int \csc^n x\cot xdx = -\frac{1}{n}\csc^n x, n \neq 0 \tag{A.91}$$

$$\int \sec x\csc xdx = \ln|\tan x| \tag{A.92}$$

Products of Trigonometric Functions and Monomials

$$\int x \cos x dx = \cos x + x \sin x \tag{A.93}$$

$$\int x \cos axdx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax \tag{A.94}$$

$$\int x^2 \cos xdx = 2x \cos x + \left(x^2 - 2\right) \sin x \tag{A.95}$$

$$\int x^2 \cos axdx = \frac{2x \cos ax}{a^2} + \frac{a^2 x^2 - 2}{a^3} \sin ax \tag{A.96}$$

$$\int x^n cosxdx = -\frac{1}{2}(i)^{n+1} \left[\Gamma(n+1, -ix) + (-1)^n \Gamma(n+1, ix)\right] \tag{A.97}$$

$$\int x^n cosaxdx = \frac{1}{2}(ia)^{1-n} \left[(-1)^n \Gamma(n+1, -iax) - \Gamma(n+1, ixa)\right] \tag{A.98}$$

$$\int x \sin xdx = -x \cos x + \sin x \tag{A.99}$$

$$\int x \sin axdx = -\frac{x \cos ax}{a} + \frac{\sin ax}{a^2} \tag{A.100}$$

$$\int x^2 \sin xdx = \left(2 - x^2\right) \cos x + 2x \sin x \tag{A.101}$$

$$\int x^2 \sin axdx = \frac{2 - a^2 x^2}{a^3} \cos ax + \frac{2x \sin ax}{a^2} \tag{A.102}$$

$$\int x^n \sin xdx = -\frac{1}{2}(i)^n \left[\Gamma(n+1, -ix) - (-1)^n \Gamma(n+1, -ix)\right] \tag{A.103}$$

Products of Trigonometric Functions and Exponentials

$$\int e^x \sin x dx = \frac{1}{2} e^x (\sin x - \cos x) \tag{A.104}$$

$$\int e^{bx} \sin ax dx = \frac{1}{a^2 + b^2} e^{bx} (b \sin ax - a \cos ax) \tag{A.105}$$

$$\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x) \tag{A.106}$$

$$\int e^{bx} \cos ax dx = \frac{1}{a^2 + b^2} e^{bx} (a \sin ax + b \cos ax) \tag{A.107}$$

$$\int x e^x \sin x dx = \frac{1}{2} e^x (\cos x - x \cos x + x \sin x) \tag{A.108}$$

$$\int x e^x \cos x dx = \frac{1}{2} e^x (x \cos x - \sin x + x \sin x) \tag{A.109}$$

Integrals of Hyperbolic Functions

$$\int \cosh ax dx = \frac{1}{a} \sinh ax \tag{A.110}$$

$$\int e^{ax} \cosh bx dx = \begin{cases} \dfrac{e^{ax}}{a^2 - b^2} [a \cosh bx - b \sinh bx] & a \neq b \\ \dfrac{e^{2ax}}{4a} + \dfrac{x}{2} & a = b \end{cases} \tag{A.111}$$

$$\int \sinh ax dx = \frac{1}{a} \cosh ax \tag{A.112}$$

$$\int e^{ax} \sinh bx dx = \begin{cases} \dfrac{e^{ax}}{a^2 - b^2} [-b \cosh bx + a \sinh bx] & a \neq b \\ \dfrac{e^{2ax}}{4a} - \dfrac{x}{2} & a = b \end{cases} \tag{A.113}$$

$$\int e^{ax} \tanh bx dx = \begin{cases} \dfrac{e^{(a+2b)x}}{(a+2b)} {}_2F_1\left[1+\dfrac{a}{2b}, 1, 2+\dfrac{a}{2b}, -e^{2bx}\right] \\ \qquad -\dfrac{1}{a} e^{ax} {}_2F_1\left[\dfrac{a}{2b}, 1, 1E, -e^{2bx}\right] & a \neq b \\ \dfrac{e^{ax} - 2\tan^{-1}[e^{ax}]}{a} & a = b \end{cases} \tag{A.114}$$

$$\int \tanh ax \, dx = \frac{1}{a} \ln \cosh ax \tag{A.115}$$

$$\int \cos ax \cosh bx dx = \frac{1}{a^2+b^2}\left[a \sin ax \cosh bx + b \cos ax \sinh bx\right] \tag{A.116}$$

$$\int \cos ax \sinh bx dx = \frac{1}{a^2+b^2}\left[b \cos ax \cosh bx + a \sin ax \sinh bx\right] \tag{A.117}$$

$$\int \sin ax \cosh bx dx = \frac{1}{a^2+b^2}\left[-a \cos ax \cosh bx + b \sin ax \sinh bx\right] \tag{A.118}$$

$$\int \sin ax \sinh bx dx = \frac{1}{a^2+b^2}\left[b \cosh bx \sin ax - a \cos ax \sinh bx\right] \tag{A.119}$$

$$\int \sinh ax \cosh ax dx = \frac{-2ax + \sinh 2ax}{4a} \tag{A.120}$$

$$\int \sinh ax \cosh bx dx = \frac{b \cosh bx \sinh ax - a \cosh ax \sinh bx}{b^2 - a^2} \tag{A.121}$$

Appendix B

Table of Laplace Transforms

$f(t)$	$\mathcal{L}[f(t)] = F(s)$	
1	$\dfrac{1}{s}$	(1)
$e^{at}f(t)$	$F(s-a)$	(2)
$\mathcal{U}(t-a)$	$\dfrac{e^{-as}}{s}$	(3)
$f(t-a)\mathcal{U}(t-a)$	$e^{-as}F(s)$	(4)
$\delta(t)$	1	(5)
$\delta(t-t_0)$	e^{-st_0}	(6)
$t^n f(t)$	$(-1)^n \dfrac{d^n F(s)}{ds^n}$	(7)
$f'(t)$	$sF(s) - f(0)$	(8)
$f^n(t)$	$s^n F(s) - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0)$	(9)

$\int_0^t f(x)g(t-x)dx$	$F(s)G(s)$	(10)
$t^n \ (n \in \mathbb{Z})$	$\dfrac{n!}{s^{n+1}}$	(11)
$t^x \ (x \geq -1 \in \mathbb{R})$	$\dfrac{\Gamma(x+1)}{s^{x+1}}$	(12)
$\sin kt$	$\dfrac{k}{s^2+k^2}$	(13)
$\cos kt$	$\dfrac{s}{s^2+k^2}$	(14)
e^{at}	$\dfrac{1}{s-a}$	(15)
$\sinh kt$	$\dfrac{k}{s^2-k^2}$	(16)
$\cosh kt$	$\dfrac{s}{s^2-k^2}$	(17)
$\dfrac{e^{at}-e^{bt}}{a-b}$	$\dfrac{1}{(s-a)(s-b)}$	(18)
$\dfrac{ae^{at}-be^{bt}}{a-b}$	$\dfrac{s}{(s-a)(s-b)}$	(19)
te^{at}	$\dfrac{1}{(s-a)^2}$	(20)
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$	(21)
$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$	(22)
$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$	(23)
$e^{at}\sinh kt$	$\dfrac{k}{(s-a)^2-k^2}$	(24)

$$e^{at}\cosh kt \qquad \frac{s-a}{(s-a)^2-k^2} \tag{25}$$

$$t\sin kt \qquad \frac{2ks}{(s^2+k^2)^2} \tag{26}$$

$$t\cos kt \qquad \frac{s^2-k^2}{(s^2+k^2)^2} \tag{27}$$

$$t\sinh kt \qquad \frac{2ks}{(s^2-k^2)^2} \tag{28}$$

$$t\cosh kt \qquad \frac{s^2-k^2}{(s^2-k^2)^2} \tag{29}$$

$$\frac{\sin at}{t} \qquad \arctan\frac{a}{s} \tag{30}$$

$$\frac{1}{\sqrt{\pi t}}e^{-a^2/4t} \qquad \frac{e^{-a\sqrt{s}}}{\sqrt{s}} \tag{31}$$

$$\frac{a}{2\sqrt{\pi t^3}}e^{-a^2/4t} \qquad e^{-a\sqrt{s}} \tag{32}$$

$$\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right) \qquad \frac{e^{-a\sqrt{s}}}{s} \tag{33}$$

Appendix C

Summary of Methods

First Order Linear Equations

Equations of the form $y' + p(t)y = q(t)$ have the solution

$$y(t) = \frac{1}{\mu(t)}\left(C + \int \mu(s)q(s)ds\right)$$

where

$$\mu(t) = exp\left(\int_t p(s)ds\right)$$

Exact Equations

An differential equation

$$M(t, y)dt + N(t, y)dy = 0$$

is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

in which case the solution is a $\phi(t) = C$ where

$$M = \frac{\partial \phi}{\partial t}, N = \frac{\partial \phi}{\partial y}$$

$$\phi(t, y) = \int M dt + \int \left(N - \int \frac{\partial M}{\partial y} dt\right) dy$$

Integrating Factors

An integrating factor μ for the differential equation

$$M(t, y)dt + N(t, y)dy = 0$$

satisfies

$$\frac{\partial(\mu(t, y)M(t, y))}{\partial y} = \frac{\partial(\mu(t, y)N(t, y))}{\partial t}$$

If

$$P(t, y) = \frac{M_y - N_t}{N}$$

is only a function of t (and not of y) then $\mu(t) = e^{\int P(t)dt}$ is an integrating factor.If

$$Q(t, y) = \frac{N_t - M_y}{M}$$

is only a function of y (and not of t) then $\mu(t) = e^{\int Q(t)dt}$ is an integrating factor.

Homogeneous Equations

An equation is homogeneous if has the form

$$y' = f(y/t)$$

To solve a homogeneous equation, make the substitution $y = tz$ and rearrange the equation; the result is separable:

$$\frac{dz}{F(z) - z} = \frac{dt}{t}$$

Bernoulli Equations

A Bernoulli equation has the form

$$y'(t) + p(t)y = q(t)y^n$$

for some number n. To solve a Bernoulli equation, make the substitution

$$u = y^{1-n}$$

The resulting equation is linear and

$$y(t) = \left[\frac{1}{\mu}\left(C + \int \mu(t)(1-n)q(t)dt\right)\right]^{1/(1-n)}$$

where

$$\mu(t) = \exp\left((1-n)\int p(t)dt\right)$$

Second Order Homogeneous Linear Equation with Constant Coefficients

To solve the differential equation

$$ay'' + by' + cy = 0$$

find the roots of the characteristic equation

$$ar^2 + br + c = 0$$

If the roots (real or complex) are distinct, then

$$y = Ae^{r_1 t} + Be^{r_2 t}$$

If the roots are repeated then

$$y = (A + Bt)e^{rt}$$

Method of Undetermined Coefficients

To solve the differential equation

$$ay'' + by' + cy = f(t)$$

where $f(t)$ is a polynomial, exponential, or trigonometric function, or any product thereof, the solution is

$$y = y_H + y_P$$

where y_H is the complete solution of the homogeneous equation

$$ay'' + by' + cy = 0$$

To find y_P make an educated guess based on the form form of $f(t)$. The educated guess should be the product

$$y_P = P(t)S(t)E(t)$$

where $P(t)$ is a polynomial of the same order as in $f(t)$. $S(t) = r^n(A \sin rt + B \cos rt)$ is present only if there are trig functions in rt in $f(t)$, and n is the multiplicity of r as a root of the characteristic equation ($n = 0$ if r is not a root). $E(t) = r^n e^{rt}$ is present only if there is an exponential in rt in $f(t)$. If $f(t) = f_1(t) + f_2(t) + \cdots$ then solve each of the equations

$$ay'' + by' + cy = f_i(t)$$

separately and add all of the particular solutions together to get the complete particular solution.

General Non-homogeneous Linear Equation with Constant Coefficients

To solve

$$ay'' + by' + cy = f(t)$$

where a, b, c are constants for a general function $f(t)$, the solution is

$$y = Ae^{r_1 t} + Be^{r_1 t} \int_t e^{r_2 - r_1} s ds + \frac{e^{r_1 t}}{a} \int_t e^{r_2 - r_1} s \int_s e^{-r_2 u} f(u) du ds$$

where r_1 and r_2 are the roots of $ar^2 + br + c = 0$.

An alternative method is to factor the equation into the form

$$(D - r_1)(D - r_2)y = f(t)$$

and make the substitution

$$z = (D - r_2)y$$

This reduces the second order equation in y to a first order linear equation in z. Solve the equation

$$(D - r_1)z = f(t)$$

for z, then solve the equation

$$(D - r_2)y = z$$

for y once z is known.

Method of Reduction of Order

If one solution y_1 is known for the differential equation

$$y'' + p(t)y' + q(t)y = 0$$

then a second solution is given by

$$y_2(t) = y_1(t) \int \frac{W(y_1, y_2))(t)}{y_1(t)^2} dt$$

where the Wronskian is given by Abel's formula

$$W(y_1, y_2)(t) = Cexp\left(-\int p(s)ds\right)$$

Method of Variation of Parameters

To find a particular solution to

$$y'' + p(t)y' + q(t)y = r(t)$$

when a pair of linearly independent solutions to the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0$$

are already known,

$$y_p = -y_1(t) \int_t \frac{y_2(s)r(s)}{W(y_1, y_2)(s)} ds + y_2(t) \int_t \frac{y_1(s)r(s)}{W(y_1, y_2)(s)} ds$$

Power Series Solution

To solve

$$y'' + p(t)y' + q(t)y = g(t)$$

expand y, p, q and g in power series about ordinary (non-singular) points and determine the coefficients by applying linear independence to the powers of t.

To solve

$$a(t)y'' + b(t)y' + c(t)y = g(t)$$

about a point t_0 where $a(t) = 0$ but the limits $b(t)/a(t)$ and $c(t)/a(t)$ exist as $t \to 0$ (a regular singularity), solve the indicial equation

$$r(r-1) + rp_0 + q_0 = 0$$

for r where $p_0 = \lim_{t\to 0} b(t_0)/a(t_0)$ and and $q_0 = \lim_{t\to 0} c(t_0)/a(t_0)$. Then one solution to the homogeneous equation is

$$y(t) = (t-t_0)^r \sum_{k=0}^{\infty} c_k (t-t_0)^k$$

for some unknown coefficients c_k.Determine the coefficients by linear independance of the powers of t. The second solution is found by reduction of order and the particular solution by variation of parameters.

Method of Frobenius

To solve

$$(t-t_0)^2 y'' + (t-t_0)p(t)y' + q(t)y = 0$$

where p and q are analytic at t_0, let $p_0 = p(0)$ and $q_0 = q(0)$ and find the roots $\alpha_1 \geq \alpha_2$ of

$$\alpha^2 + (p_0 - 1)\alpha + q_0 = 0$$

Define $\Delta = \alpha_1 - \alpha_2$. Then for some unknowns c_k, a first solution is

$$y_1(t) = (t-t_0)^{\alpha_1} \sum_{k=0}^{\infty} c_k (t-t_0)^k$$

If $\Delta \in \mathbb{R}$ is not an integer or the roots are complex,

$$y_2(t) = (t-t_0)^{\alpha_2} \sum_{k=0}^{\infty} a_k (t-t_0)^k$$

If $\alpha_1 = \alpha_2 = \alpha$, then $y_2 = a y_1(t) \ln|t-t_0| + (t-t_0)^{\alpha} \sum_{k=0}^{\infty} a_k (t-t_0)^k$

If $\Delta \in \mathbb{Z}$, then $y_2 = a y_1(t) \ln|t-t_0| + (t-t_0)^{\alpha_2} \sum_{k=0}^{\infty} a_k (t-t_0)^k$

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Live
Long
and
have
a
good
Summer


CALCULUS
850th edition
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solves life's problems!
Fun!
Recommended